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STRUCTURAL AND SELF-SIMILAR PROPERTIES OF REPRESENTATIONS OF ONE CLASS OF FRACTAL FUNCTIONS AND DISTRIBUTIONS OF THEIR VALUES

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Abstract. We consider the Q_2-representation of numbers from the interval $[0;1]$, defined by parameters $q_0, q_1 \in (0;1)$, and expansion of an arbitrary number $x \in [0;1]$ by the series

$$x = \alpha_1 q_{1-\alpha_1} + \sum_{k=2}^{\infty} (\alpha_k q_{1-\alpha_k} \prod_{j=1}^{k-1} q_{\alpha_j}(x)) \equiv \Delta^{Q_2}_{\alpha_1\alpha_2\ldots\alpha_n\ldots},$$

where $\alpha_k \in \{0,1\} \equiv A$, $q_0 + q_1 = 1$. We study structural, local, and global topological, metric, and fractal properties of the function defined by the equality

$$f_{\overline{\varphi}}(x = \Delta^{Q_2}_{\alpha_1\alpha_2\alpha_3\ldots\alpha_n\alpha_{n+1}}) = \Delta^{Q_2}_{\varphi_1(\alpha_1,\alpha_2)\varphi_2(\alpha_2,\alpha_3)\ldots\varphi_n(\alpha_n,\alpha_{n+1})\ldots},$$

where $\overline{\varphi} = (\varphi_n)$ is a given sequence of functions $(\varphi : (0;1)^2 \to (0;1))$. For a random variable $Y = F(X)$, where X is a random variable with a given distribution, we investigate the Lebesgue structure and spectral properties.

Introduction

One of the scale-invariant continuous generalizations of the classical binary representation [3] of numbers in the interval $[0,1]$ is the two-symbol $A \equiv \{0;1\}$ multibasic Q_2^*-representation [2] that is defined by a positive infinite numerical sequence

$$(q_{0k}) = (q_{01}, q_{02}, ..., q_{0k}, ...),\ 0 < q_{0k} < 1,\ \prod_{k=1}^{\infty} \max\{q_{0k}, 1 - q_{0k}\} = 0,$$

and is given by

$$x = \alpha_1 q_{1_{-\alpha_1},1} + \sum_{k=2}^{\infty} (\alpha_k q_{1_{-\alpha_k},k} \prod_{j=1}^{k-1} q_{\alpha_j,j}) \equiv \Delta^{Q_2^*}_{\alpha_1\alpha_2\ldots\alpha_k\ldots}, \tag{1}$$

where $q_{1k} \equiv 1 - q_{0k}$, $(\alpha_k) \in L = A \times A \times ...$ (further, $\alpha_k q_{1-\alpha_k,k} \equiv \beta_{\alpha_k,k}$).

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The abbreviated (symbolic) notation $\Delta^{Q_2^*}_{\alpha_1\alpha_2\ldots\alpha_k\ldots}$ in identity (1) is called the Q_2^*-representation of x. At the same time, α_k is called the k-th digit of this representation. Most numbers in the interval $[0,1]$ have a unique Q_2^*-representation, we call them Q_2^*-*unary* numbers. And only the numbers of a countable set have two Q_2^*-representation and are called Q_2^*-*binary* numbers. Their representations have the following form: $\Delta^{Q_2^*}_{c_1c_2\ldots c_{m-1}1(0)}$ and $\Delta^{Q_2^*}_{c_1c_2\ldots c_{m-1}0(1)}$. For Q_2^*-unary numbers, the function $\alpha_k = \alpha_k(x)$ is well-defined, but for Q_2^*-binary numbers it is not so.

If $q_{0k} = q_0$ for any $k \in N$, then the Q_2^*-representation coincides with the Q_2-representation, and if $q_{0k} = \frac{1}{2}\ \forall k \in N$, then the Q_2^*-representation is the classical binary representation. The geometry of the Q_2^*-representation was described in [3] and depends on properties of its cylindrical sets.

A cylinder of rank m with base $c_1c_2\ldots c_m$ is called the set $\Delta^{Q_2^*}_{c_1c_2\ldots c_m}$ of all points $x \in [0,1]$, which have Q_2^*-representation $\Delta^{Q_2^*}_{c_1c_2\ldots c_m\alpha_{m+1}\alpha_{m+2}\ldots}$, $\alpha_{m+i} \in A$. Cylinder $\Delta^{Q_2^*}_{c_1c_2\ldots c_m}$ is the interval $[a,b]$ with endpoints

$$a = \beta_{c_1,1} + \sum_{k=2}^{\infty}(\beta_{c_k,k}\prod_{j=1}^{k-2} q_{c_j,j}), \quad b = a + \prod_{j=1}^{m} q_{c_j,j}.$$

We consider two Q_2^*-representations of numbers defined by the sequences (q_{0k}) and (g_{0k}), respectively (we will call the second G_2^*-representation), and the sequence $\overline{\varphi} \equiv (\varphi_k)$ of finite functions $\varphi_k : A \times A \to A$.

We consider the function defined by

$$f_{\overline{\varphi}}(\Delta^{Q_2^*}_{\alpha_1\alpha_2\ldots\alpha_k\ldots}) = \Delta^{G_2^*}_{\varphi_1(\alpha_1,\alpha_2)\varphi_2(\alpha_2,\alpha_3)\ldots\varphi_k(\alpha_k,\alpha_{k+1})\ldots}, \tag{2}$$

which maps $[0;1]$ onto $[0;1]$. Function $f_{\overline{\varphi}}$ is well-defined if and only if the expression (2) takes the same value for distinct Q_2^*-representations of binary numbers. It is easy to indicate an example of such sets $c_1c_2\ldots c_k$, where the inequality $f_{\overline{\varphi}}(\Delta^{Q_2^*}_{c_1c_2\ldots c_k0(1)}) \neq f_{\overline{\varphi}}(\Delta^{Q_2^*}_{c_1c_2\ldots c_k1(0)})$ holds. It is possible to ensure that the function is well-defined by agreeing to use only the representation that has period (0). Note that the number 1 has the unique Q_2^*-representation $\Delta^{Q_2^*}_{(1)}$.

There are exactly 16 distinct functions φ_k: $A \times A \to A$.

It is convenient to associate the function φ_k with a matrix $\begin{pmatrix} a_{00}^{(k)} & a_{01}^{(k)} \\ a_{10}^{(k)} & a_{11}^{(k)} \end{pmatrix}$, whose elements are either 0 or 1", namely, $a_{ij}^{(k)} = \varphi_k(i,j) \in A$. Then each

of the functions $f_{\overline{\varphi}}$ can be associated with a sequence of matrices

$$M_k = \begin{pmatrix} a_{00}^{(k)} & a_{01}^{(k)} \\ a_{10}^{(k)} & a_{11}^{(k)} \end{pmatrix}.$$

Next, we define a group operation $*$ on matrices

$$M * M' = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} * \begin{pmatrix} a'_{00} & a'_{01} \\ a'_{10} & a'_{11} \end{pmatrix} = \begin{pmatrix} d_{00} & d_{01} \\ d_{10} & d_{11} \end{pmatrix},$$

where $d_{ij} = |a_{ij} - a'_{ij}|$. As a result, we obtain the commutative group of matrices. Indeed, the identity element is the 2×2 zero matrix (the trivial function $f_{\overline{\varphi}} = 0$ corresponds to this matrix with a constant sequence $\overline{\varphi}$). Each function in this group is self-inverse. The operation on matrices generates a group of functions $\varphi : A \times A \to A$, and this induces a group S of functions $f_{\overline{\varphi}}$ for a constant sequence of functions $\overline{\varphi}$. At the same time, the result of the operation on a pair of dual functions φ and $\widetilde{\varphi} = 1 - \varphi$ is the trivial function $f_{\overline{\varphi}} = 1$, which corresponds to a matrix with elements equal to 1. Each function in this group is symmetric (inverse) to itself.

There are 8 pairs of dual functions among the 16 functions φ, that have at first glance "similar" properties and are determined with accuracy up to "symmetry" of digits. These are f_φ functions generated by the following functions:

$$\varphi_{[1]}(i,j) = \begin{cases} 0, \text{if } i \cdot j = 0, \\ 1, \text{if } i \cdot j \neq 0; \end{cases} \qquad \varphi_{[2]}(i,j) = \begin{cases} 0, \text{if } i(i+j) \neq 1, \\ 1, \text{if } i(i+j) = 1; \end{cases}$$

$$\varphi_{[3]}(i,j) = \begin{cases} 0, \text{if } i(i+j) = 0, \\ 1, \text{if } i(i+j) \neq 0; \end{cases} \quad \varphi_{[4]}(i,j) = \begin{cases} 0, \text{if } j(i+j) \neq 1, \\ 1, \text{if } j(i+j) = 1; \end{cases}$$

$$\varphi_{[5]}(i,j) = \begin{cases} 0, \text{if } j(i+j) = 0, \\ 1, \text{if } j(i+j) \neq 0; \end{cases} \quad \varphi_{[6]}(i,b) = \begin{cases} 0, \text{if } i = j, \\ 1, \text{if } i \neq j; \end{cases}$$

$$\varphi_{[7]}(i,j) = \begin{cases} 0, \text{if } i + j = 0, \\ 1, \text{if } i + j \neq 0; \end{cases}$$

or $\varphi_{[1]}(i,j) = i \cdot j$, $\varphi_{[2]}(i,j) = |i-j|i$, $\varphi_{[3]}(i,j) = i$, $\varphi_{[4]}(i,j) = |i-j|j$,

$\varphi_{[5]}(i,j) = j$, $\varphi_{[6]}(i,j) = |i-j|$, $\varphi_{[7]}(i,j) = i \cdot j + |i-j|$,
and their doubles.

1. Individual cases

Lemma 1. *For the function $f_{\overline{\varphi}}$ to be constant $c = \Delta^{G_2^*}_{c_1 c_2 \ldots c_n \ldots}$ it is necessary and sufficient that $\varphi_n(a,b) = c_n$ for all $n \in N$ and $(a,b) \in A_2 \times A_2$.*

Proof. Necessity. Since the equality $f_{\overline{\varphi}}(x) = \Delta^{G_2^*}_{c_1c_2\ldots c_n\ldots}$ holds for any $x \in [0,1]$, i.e.,

$$\begin{cases} \varphi_1(\alpha_1(x), \alpha_2(x)) = c_1 \text{ for } \forall(\alpha_1, \alpha_2) \in A \times A; \\ \varphi_2(\alpha_2(x), \alpha_3(x)) = c_2 \text{ for } \forall(\alpha_2, \alpha_3) \in A \times A; \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \varphi_n(\alpha_n(x), \alpha_{n+1}(x)) = c_n \text{ for } \forall(\alpha_n, \alpha_{n+1}) \in A \times A; \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \end{cases}$$

we get the identity $\varphi_n(a,b) = c_n$ for any $(a,b) \in A \times A$.

Sufficiency. If $\varphi_n(a,b) = c_n$ for any $n \in N$ and for each pair $(a,b) \in A \times A$, then

$$\begin{aligned} f_{\overline{\varphi}}(x = \Delta^{Q_2^*}_{\alpha_1(x)\alpha_2(x)\ldots\alpha_n(x)\ldots}) &= \Delta^{G_2^*}_{\varphi_1(\alpha_1,\alpha_2)\varphi_2(\alpha_2,\alpha_3)\ldots\varphi_n(\alpha_n,\alpha_{n+1})\ldots} = \\ &= \Delta^{G_2^*}_{c_1c_2\ldots c_n\ldots} = c. \end{aligned}$$

The sufficiency and the entire lemma are proven. □

Corollary 1. *There exists a continuum set S of continuous functions $f_{\overline{\varphi}}$ that are constants. Moreover, for any $c \in [0;1]$ there exists $f_{\overline{\varphi}} \in S$ such that $f_{\overline{\varphi}}(x) = c$.*

Definition 1. A function $f_{\overline{\varphi}}$ such that $\varphi_n(a,b) = a$ for any $n \in N$ is called quasi-identical (Q_2^*-G_2^*)-transformation of the interval $[0;1]$.

Remark 1. The argument for choosing the term «quasi-identical (Q_2^*-G_2^*)-transformation» is the fact that $f_{\overline{\varphi}}(x) = x$ provided that $G_2^* = Q_2^*$, i.e., $q_{0n} = g_{0n}$ for any $n \in N$.

Lemma 2. *If $\varphi_n(a,b) = a$ for any $n \in N$, i.e., $f_{\overline{\varphi}}(\Delta^{Q_2^*}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}) = \Delta^{G_2^*}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}$, then $f_{\overline{\varphi}}$ is a continuous, strictly increasing distribution function F_ξ of a random variable $\xi = \Delta^{Q_2^*}_{\xi_1\xi_2\ldots\xi_n\ldots}$. The digits (ξ_n) of its Q_2^*-representation are independent and have distributions*

$$P\{\xi_n = 0\} = g_{0n}, \quad P\{\xi_n = 1\} = g_{1n}.$$

The function F_ξ is either absolutely continuous or singular.

Indeed, the expression for the distribution function F_ξ of the random variable ξ has a G_s^*-expansion (representation). A more detailed justification can be found in the article [13], where other properties of this function, such as differential property, were also studied. The function belongs to the pure Lebesgue type, and the criteria for absolute continuity and singularity are established. In general, the differential properties of the function F_ξ are quite complex.

2. Sets of value class functions **S**

Lemma 3. *If in the sequence (φ_n) the function φ_m is a constant i, then the set of values of the function $f_{\overline{\varphi}} \in S$ is a subset of the union of m-th rank cylinders, namely,*

$$E_{f_{\overline{\varphi}}} \subset \bigcup_{c_1 \in A_2} \bigcup_{c_2 \in A_2} \dots \bigcup_{c_{m-1} \in A_2} \Delta^{G_2^*}_{c_1 c_2 \dots c_{m-1} i}. \tag{3}$$

Proof. Let (φ_n) be a sequence that satisfies the conditions of the statement, and let $f_{\overline{\varphi}}$ be its corresponding function. Since $\varphi_m(a,b) = i$ for any pair (a,b), then m-th digit is the G_2^*-representation of the values of the function $f_{\overline{\varphi}}$ is not equal to $1-i$. This implies the inclusion (3). □

Corollary 2. *If there is an infinite subsequence of constants among the functions of the sequence (φ_n), then the set of values of $f_{\overline{\varphi}}$ is a subset of a Cantor-type set $C[G_2^*, V_k]$, where*

$$V_k = \begin{cases} A_2, & \text{if } \varphi_k(a;b) - \text{is not a constant;} \\ \{j_k\}, & \text{if } \varphi_k(a,b) = j_k. \end{cases}$$

3. Continuity of functions of class S

Lemma 4. *Every function $f_{\overline{\varphi}}$ of class S is continuous on the set Q_2^*-unary numbers.*

Proof. To prove continuity at a Q_2^*-unary point $x = \Delta^{Q_2^*}_{\alpha_1\alpha_2\alpha_3\dots\alpha_n\dots}$ of the interval $[0;1]$, we must show that

$$\lim_{x' \to x} |f_{\overline{\varphi}}(x') - f_{\overline{\varphi}}(x)| = 0. \tag{4}$$

Since $x' \neq x$, there exists a number m such that $\alpha_m(x') \neq \alpha_m(x)$ and $\alpha_i(x') = \alpha_i(x)$ for all $i < m$. Then condition $x' \to x$ is equivalent to $m \to \infty$. Without loss of generality, let's assume that $\alpha_m(x) = 1$, which implies $\alpha_m(x') = 0$.

Consider the difference of the values of $f_{\overline{\varphi}}$ at points x and x':

$$\begin{aligned} f_{\overline{\varphi}}(x) - f_{\overline{\varphi}}(x') &= f_{\overline{\varphi}}(\Delta^{Q_2^*}_{\alpha_1\alpha_2\dots\alpha_{m-1}1\alpha_{m+1}\dots}) - f_{\overline{\varphi}}(\Delta^{Q_2^*}_{\alpha_1\alpha_2\dots\alpha_{m-1}0\alpha'_{m+1}\dots}) = \\ &= \Delta^{G_2^*}_{\varphi_1(\alpha_1,\alpha_2)\dots\varphi_{m-1}(\alpha_{m-1},1)\varphi_m(1,\alpha_{m+1})\dots} - \\ &- \Delta^{G_2^*}_{\varphi_1(\alpha_1,\alpha_2)\dots\varphi_{m-1}(\alpha_{m-1},0)\varphi_m(0,\alpha'_{m+1})\dots} = \prod_{j=1}^{m-2} g_{\varphi_j(\alpha_j,\alpha_{j+1}),j} \cdot A, \end{aligned}$$

where $|A| \le 1$. Then $\lim\limits_{m\to\infty} \prod\limits_{j=1}^{m-2} g_{\varphi_j(\alpha_j,\alpha_{j+1}),j} = 0$. Consequently,

$$A \prod_{j=1}^{m-2} g_{\varphi_j(\alpha_j,\alpha_{j+1}),j} \to 0,$$

as $m \to \infty$ that proves the equality $\lim\limits_{x'\to x} |f_{\overline{\varphi}}(x') - f_{\overline{\varphi}}(x)| = 0$. □

Theorem 1. *For the function $y = f_{\overline{\varphi}}(x)$ to be continuous, it is necessary and sufficient that its values at two distinct representations Q_2^*-binary numbers coincide, i.e.,*

$$f_{\overline{\varphi}}(\Delta^{Q_2^*}_{c_1\ldots c_m 0(1)}) = f_{\overline{\varphi}}(\Delta^{Q_2^*}_{c_1\ldots c_m 1(0)}), \tag{5}$$

for any sequence of digits $(c_1, ..., c_m)$ consisting of zeros and ones.

Proof. Let $x_0 = \Delta^{Q_2^*}_{c_1\ldots c_m 0(1)} = \Delta^{Q_2^*}_{c_1\ldots c_m 1(0)}$ be a Q_2^*-binary point.
1. Necessity. If a function $f_{\overline{\varphi}}$ is continuous at a point x_0, i.e.,

$$\lim_{x\to x_0} f_{\overline{\varphi}}(x) = f_{\overline{\varphi}}(x_0),$$

then it is continuous from the left and from the right at this point. Consider the sequence $x_n = \Delta^{Q_2^*}_{c_1\ldots c_m 0\underbrace{1\ldots 1}_{n}(0)} \to x_0 - 0$ $(n \to \infty)$. Then

$$\lim_{n\to\infty} f_{\overline{\varphi}}(x_n) = f_{\overline{\varphi}}(x_0) = f_{\overline{\varphi}}(\Delta^{Q_2^*}_{c_1\ldots c_m 0(1)}).$$

Now consider the sequence $\overline{x}_n = \Delta^{Q_2^*}_{c_1\ldots c_m \underbrace{0\ldots 0}_{n}1(0)} \to x_0 + 0$. Then

$$\lim_{n\to\infty} f_{\overline{\varphi}}(\overline{x}_n) = \lim_{n\to\infty} f_{\overline{\varphi}}(\Delta^{Q_2^*}_{c_1\ldots c_m \underbrace{0\ldots 0}_{n}1(0)}) = f_{\overline{\varphi}}(x_0) = f_{\overline{\varphi}}(\Delta^{Q_2^*}_{c_1\ldots c_m 1(0)}).$$

Thus, equality (5) holds.

2. Sufficiency. Now we will prove that the continuity of $f_{\overline{\varphi}}$ at the point x_0 follows from the condition

$$f_{\overline{\varphi}}(x_0) = f_{\overline{\varphi}}(\Delta^{Q_2^*}_{c_1\ldots c_m 0(1)}) = f_{\overline{\varphi}}(\Delta^{Q_2^*}_{c_1\ldots c_m 1(0)}) \tag{6}$$

First, we prove left-hand continuity at the point x_0. To do this, consider a number $x < x_0$ that is sufficiently close to x_0, spesifically $x = \Delta^{Q_2^*}_{c_1\ldots c_m 0\underbrace{1\ldots 1}_{n}\alpha_{m+n+2}\alpha_{m+n+3}\ldots}$. The modulus of the difference is

$$\delta \equiv |f_{\overline{\varphi}}(x_0) - f_{\overline{\varphi}}(x)| = A_n |\omega^{m+n+1}(f_{\overline{\varphi}}(x)) - \omega^{m+n+1}(f_{\overline{\varphi}}(x_0))|,$$

where $A_n = g_{\varphi_m(c_m,0),m} g_{\varphi_{m+1}(0,1),m+1} \prod\limits_{j=1}^{m-1} g_{\varphi_j(c_j,c_{j+1}),j} \prod\limits_{i=1}^{n} g_{\varphi_{m+1+i}(1,1),m+1+i}$.

Since $x \to x_0 \Leftrightarrow n \to \infty$ is equivalent to $n \to \infty$ and $A_n \to 0$ as $n \to \infty$, then $\delta \to \infty$. This proves the left-hand continuity of the function $f_{\overline{\varphi}}$ at the point x_0.

Similar considerations justify the right-hand continuity of the function at the point x_0. In this case, an arbitrary $x > x_0$ sufficiently close to x_0 is considered, namely: $x = \Delta^{Q_2^*}_{c_1 \ldots c_m 1 \underbrace{0 \ldots 0}_{n} \alpha_{m+n+2} \alpha_{m+n+3} \ldots}$. □

Theorem 2. *If the function $f_{\overline{\varphi}}$ is continuous at every point of the interval $[0, 1]$, then it is either an identity transformation, an inversor, or a constant.*

Proof. Since $f_{\overline{\varphi}}$ is continuous on $[0, 1]$, then according to the previous theorem, it is continuous at every Q_2^*-binary point. This means that

$$f_{\overline{\varphi}}(\Delta^{Q_2^*}_{\alpha_1 \alpha_2 \ldots \alpha_{n-1} 0(1)}) = f_{\overline{\varphi}}(\Delta^{Q_2^*}_{\alpha_1 \alpha_2 \ldots \alpha_{n-1} 1(0)}), n \in N.$$

It follows that for any $n \in N$ the equalities hold

$$\varphi_n(0,1) = \varphi_n(1,0) = \varphi_n(0,0) = \varphi_n(1,1) \text{ or}$$
$$\varphi_n(0,1) \neq \varphi_n(1,0) = \varphi_n(0,0) \neq \varphi_n(1,1).$$

In the first case, $\varphi(a, b) = Const$. In the second case, each of the functions φ_n is given by a matrix $\begin{pmatrix} i & i \\ j & j \end{pmatrix}$, where $\varphi_n(0,1) = \varphi_n(0,0) = i$, $\varphi_n(1,0) = \varphi_n(1,1) = j$, $i, j \in A$. Such functions correspond to the identity transformation $y = f_{\overline{\varphi}}(x) = x$ and $y = I(x)$. Therefore, the set of continuous functions of class S is exhausted by constants, the identity transformation, and the inversor. □

Corollary 3. *Functions of class S are continuous on the set of Q_2-unary numbers and discontinuous on the set of Q_2-binary numbers, unless they are continuous everywhere.*

4. Level sets of functions of class S

Let $q_{0k} = g_{0k} = q_0$. Recall that a level set of a function f for a given value $y_0 \in E_f$ is the set of all points of the domain that are mapped to y_0. Formally, this can be written as $f^{-1}(y_0) \equiv \{x : f(x) = y_0\}$.

Theorem 3. *The level set of functions $f_{\overline{\varphi}}$, where $\varphi_n = \varphi$ for all $n \in N$, is either a finite or continuous set. Moreover,*
1) if $E_{f_{\overline{\varphi}}} = \{i\}, i \in A$, then the set $f^{-1}_{\overline{\varphi}}(y_0)$ has cardinality continuum;
2) if $E_{f_{\overline{\varphi}}} = [0; 1]$, then the set $f^{-1}_{\overline{\varphi}}(y_0)$ is a singleton or consists of only two points x_0 and $I(x_0)$, where $I(\Delta^{Q_2}_{\alpha_1 \ldots \alpha_n \ldots}) = \Delta^{Q_2}_{[1-\alpha_1] \ldots [1-\alpha_n] \ldots}$;
3) if $E_{f_{\overline{\varphi}}} = [Q_2, V]$, then the set $f^{-1}_{\overline{\varphi}}(y_0)$ is a finite or continuum set.

Proof. We will prove this theorem case by case.

1) By the definition of the function, if $E_{f_{\overline{\varphi}}} = \{y_0\}$ then $f_{\overline{\varphi}}(x) = y_0$ for all $x \in [0,1]$. Therefore, the level set $f^{-1}_{\overline{\varphi}}(y_0)$ is the entire interval $[0;1]$, that has cardinality continuum.

2) If $E_{f_{\overline{\varphi}}} = [0;1]$, then the functions $f_{\overline{\varphi}}$ is exhausted by the identity function $f_{\overline{\varphi_{[6]}}}$, $y = x$, the left-shift operator $w(x)$ and their inversors. For the identity function $y = x$, it is obvious that every image has a unique preimage. For the left-shift operator $y = w(x)$, the function is defined as: $\forall x \in [0;1)$: $f_{\overline{\varphi}}(x = \Delta^{Q_2}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}) = w(x) = y = \Delta^{Q_2}_{\alpha_2\ldots\alpha_n\ldots}$. For any given output y, the set of preimages is $f^{-1}_{\overline{\varphi}} = \{\Delta^{Q_2}_{1\alpha_2\ldots\alpha_n\ldots}, \Delta^{Q_2}_{0\alpha_2\ldots\alpha_n\ldots}\}$. Thus, the level sets of the left-shift operator consist of two points. In the case of $f_{\overline{\varphi_{[6]}}}$, due to the function's symmetry with respect to the inversor meaning that $f_{\overline{[\varphi_6]}}(x) = f_{\overline{[\varphi_6]}}(I(x))$, each Q_2-unary level consists of two points. Each Q_2-binary level consists of a single point because we exclude representations that contain an infinite sequence of 1 (i.e., we avoid representations with a repeating period (1)).

3) If $E_{f_{\overline{\varphi}}} = [Q_2, V]$, then the function defining $f_{\overline{\varphi}}$ is given by:

$$\varphi(\alpha_k, \alpha_{k+1}) = \varphi(\overline{\alpha}_k, \alpha_{k+1}) = \varphi(\overline{\alpha}_k, \overline{\alpha}_{k+1}) = i,\ \varphi(\alpha_k, \overline{\alpha}_{k+1}) = 1 - i.$$

Then for an arbitrary $y = \Delta^{Q_2}_{c_1c_2\ldots c_m\ldots} \in E_{f_{\overline{\varphi}}}$, the level set has form

$$f^{-1}_{\overline{\varphi}}(y) = \{x := \Delta^{Q_2}_{a_1a_2\ldots a_m\ldots}, \varphi(a_m, a_{m+1}) = c_m\}.$$

For every $x \in f^{-1}_{\overline{\varphi}}(y)$ we establish a correspondence to a number $z \in [0;1]$, where $z = \sum\limits_{k=1}^{\infty} \frac{z_k}{2^k}$, according to the following rule:

$$\begin{cases} z_n = 0, \text{ if for any } a_n \text{ we have } \varphi(a_n, a_{n+1}) = i, \\ z_n = 1, \text{ if only for } \overline{a}_n \text{ we have } \varphi(\overline{a}_n, a_{n+1}) = 1 - i. \end{cases}$$

In this way, the correspondence between the level set $f^{-1}_{\overline{\varphi}}$ and the unit interval $[0,1]$ is established. Therefore, the level set $f^{-1}_{\overline{\varphi}}(y)$ is continuum set. Examples of finite level sets are provided in the article [8]. The theorem is proved. □

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