VORONOÏ'S IMPACT ON MODERN SCIENCE

Proceedings
of the Sixth International Conference
on Analytic Number Theory
and Spatial Tessellations

Volume 1

National Pedagogical Dragomanov University

VORONOÏ'S IMPACT ON MODERN SCIENCE

Proceedings
of the Sixth International Conference
on Analytic Number Theory
and Spatial Tessellations

In two volumes
Volume 1

Edited by Jörn Steuding and Mykola Pratsiovytyi
Compiled by Halyna Syta

Natl. Pedagog. Dragomanov Univ. Publ.
Kyiv ∘ 2018

UDC 511.32, 511.33, 511.41, 511.48, 511.52, 517.521.8
MSC2010 00B25, 03G10, 11A25, 11B37, 11D72, 11F30, 11J91, 11K65, 11L05, 11M06, 11M26, 11M41, 11N25, 11N37, 11N60, 33B10, 40A15, 42B08, 60F17

Voronoï's Impact on Modern Science. Proceedings of the Sixth International Conference on Analytic Number Theory and Spatial Tessellations : In two volumes / Jörn Steuding and Mykola Pratsiovytyi (Eds.) ; Halyna Syta (Compil.) ; Natl. Pedagog. Dragomanov Univ. — Kyiv : Natl. Pedagog. Dragomanov Univ. Publ., 2018. — Vol. 1. — xvi+178 p.

The papers in this book comprise the Part 1 of the Proceedings of the Sixth International Conference on Analytic Number Theory and Spatial Tessellations held in Kyiv (Ukraine) on September 24–28, 2018.

Referees: Yuriy Drozd, D.Sc., Corresp. Memb. NAS Ukraine
Mykola Semko, D.Sc.

Affirmed to publication by the Scientific Council
of the National Pedagogical Dragomanov University,
Minutes No. 3 of November 28, 2018

Вплив Вороного на сучасну науку. Праці Шостої міжнародної конференції з аналітичної теорії чисел і просторових мозаїк : У двох томах / Ред. : Йорн Штойдинґ, Микола Працьовитий ; упоряд. Галина Сита ; Нац. пед. ун-т ім. М. П. Драгоманова. — Київ : Вид-во Нац. пед. ун-ту ім. М. П. Драгоманова, 2018. — Т. 1. — xvi+178 с.

Статті цієї книги складають першу частину праць Шостої міжнародної конференції з аналітичної теорії чисел і просторових мозаїк, що відбулася в Києві (Україна) з 24 по 28 вересня 2018 року.

Рецензенти: Юрій Дрозд, д-р фіз.-мат. наук, чл.-кор. НАН України
Микола Семко, д-р фіз.-мат. наук

Схвалено для публікації Вченою радою
Національного педагогічного університету імені М. П. Драгоманова,
протокол № 3 від 28 листопада 2018 р.

ISBN 978-966-931-197-9
ISBN 978-966-931-198-6 (Vol. 1)

© National Pedagogical Dragomanov University, 2018

Contents

Preface **vii**

The Anniversary of Georgy Voronoï **viii**

Welcome Addresses **x**

Conference Organization Structure **xiv**

Plenary Session **1**

1 Two arithmetic properties of real exponential polynomials
Vichian Laohakosol, Pattira Ruengsinsub

24 Steinitz's numbers and diagonal constructions of discrete structures
Bogdana Oliynyk

33 The Riemann zeta-function on horizontal shifts of its complex zeros
Teerapat Srichan, Jörn Steuding

40 Multivariable averages of divisor functions
László Tóth

Section 1: Number Theory **41**

41 On combinatorial identities for Jacobsthal polynomials
Taras Goy

48 Discriminative products and multiplicative functions
Vichian Laohakosol, Nittiya Pabhapote

65 Extensions of Schwarz's theorem
Vichian Laohakosol, Angkana Sripayap

76 Dirichlet divisor problem on Gaussian integers
Andrew V. Lelechenko

87 Integer solutions of implicit linear difference equations
Vsevolod Martseniuk, Sergiy Gefter, Aleksey Piven'

96 On Riesz means for the coefficients of hybrid symmetric square L-functions
Olga Savastru

107 The partial sum of some functions under a constraint on the sum of digits of squares
Teerapat Srichan

120 On the universality for zeta-function of certain cusp forms
Adelė Vaiginytė

124 Problem of ellipse in arithmetic progressions
Pavel Varbanets, Sergey Varbanets

138 Solving a functional equation involving one multiplicative and two q-additive functions
Boonrod Yuttanan, Vichian Laohakosol, Sukrawan Mavecha

Section 2: Analysis **146**

146 On the Voronoi summability of Fourier integrals and its conjugate integrals
Liliya Boytsun, Tamara Rybnikova

159 Limit behavior of the expected Esscher transformation
Vitaliy Drozdenko

169 Corresponding N-dimensional continued fractions for N-multiple power series
Khrystyna Kuchminska

Author Index **177**

Preface

This year we have celebrated the 150th anniversary of the birth of Georgy Voronoï (born on April 28, 1868 in Zhuravka, died on November 20, 1908 in Warsaw, buried in Zhuravka).

The work of this eminent and very influential Ukrainian mathematician had a strong impact to various disciplines of mathematics and, in particular, to geometry and number theory (and their intersections with other disciplines).

In late September 2018, scientists from all around the world gathered in Kyiv for the Sixth International Conference on Analytic Number Theory and Spatial Tessellations in honour of Georgy Voronoï.

This event was hosted at the National Pedagogical Dragomanov University and had been devoted to the many facets of his research, such as Voronoi diagrams and their applications, number theory, methods of summability, and the so-called geometry of numbers.

The conference was a great success: in the language of mathematics, it was an "analytic continuation" of the former five conferences, perfectly organized by the local staff and, last not least, by Dr. Halyna Syta who again made this meeting something special. Thanks to the excellent atmosphere, the attendants (or I better say friends) may hope for a reunion in five years, the usual frequency of those "Voronoï conferences".

These proceedings present the contributions to number theory and a second volume will be dedicated to geometry. The topics range from arithmetical functions via lattice points and transcendence to zeta-functions. Most of them continue in one way or another the studies of Georgy Voronoï.

Let us always keep in mind what a great mathematician he was.

November 2018

Professor Jörn Steuding
Editor of the volume,
Chair of the Section of Number Theory

The Anniversary of Georgy Voronoï

In this collection, the papers of participants of the Sixth International Conference on Analytic Number Theory and Spatial Tessellations are compiled.

This conference, like the previous five ones, was devoted to the development and generalization of the scientific heritage of Georgy Voronoï (1868–1908), a scientist whose mathematical achievements were highly appreciated by his contemporaries, but they have been widely used in our days.

Such conferences take place in Kyiv (once in five years), because G. Voronoï was a Ukrainian by origin, his motherland was here, he was buried here according to his last will, his archive is kept in the Institute of Manuscripts of the Vernadsky National Library of Ukraine. This idea was supported by our foreign participants in 1993.

April 28, 2018 was the 150th anniversary of Georgy Voronoï. In Ukraine, this event was celebrated at the national level (in accordance to the Resolution of the Verkhovna Rada (Parliament) of Ukraine "On Celebration of Memorial Dates and Anniversaries in 2018" No. 2287-VIII of February 8, 2018). A series of events devoted to this date were organized by the National Pedagogical Dragomanov University, National Academy of Sciences of Ukraine, Kyiv Mathematical Society, Shevchenko Scientific Society and others.

The Sixth International Conference on Analytic Number Theory and Spatial Tessellations was held at the Faculty of Physics and Mathematics of the National Pedagogical Dragomanov University on September 24–28, 2018. The conference was organized jointly with the Institute of Mathematics of the National Academy of Sciences of Ukraine; Taras Shevchenko National University of Kyiv; Ukrainian International Committee of Science and Culture, NAS of Ukraine; Voronoi Diagram Research Center, Hanyang University (Seoul, Korea); Institute of Mathematics of the Polish Academy of Sciences (Warsaw, Poland).

Over 70 scientists from 18 countries of the world (Austria, Belarus, Ecuador, France, Germany, Greece, Hungary, Japan, Korea, Lithuania, the Netherlands, Poland, Spain, Sweden, Thailand, Turkey, Ukraine, United States of America) participated in the conference.

During the conference, 14 plenary lectures were delivered by Francesc Antón Castro (Ecuador), Rikard Bøgvad (Sweden), Sergiy Butenko (United States of America), Driss Essouabri (France), Vincent Icke (the Netherlands), Deok-Soo Kim (Korea), Vichian Laohakosol (Thailand),

William Mance (Poland), Bogdana Oliynyk (Ukraine), Mykola Pratsiovytyi (Ukraine), Jörn Steuding (Germany), Grygoriy Torbin (Ukraine), László Tóth (Hungary), Bastian Weiß (Austria).

The conference talks were related to research interests of Georgy Voronoï mainly. The following four sections worked at the conference:

1. Number Theory,
2. Analysis,
3. Voronoi Diagrams (Theoretical Aspects and Applications),
4. Fractal Analysis and Fractal Geometry.

The following honored guests visited the opening ceremony of the conference: First Deputy Minister of Education and Science of Ukraine Volodymyr Kovtunets, Ambassador of the Republic of Korea to Ukraine Yang-Goo Lee, Head of Section for Cooperation in Science and Education at the Embassy of the Republic of Poland in Ukraine Emilia Jasiuk, Georgy Voronoï's great-granddaughters Maryna Kosenko (Kyiv) and Irina Voronaya (Vitebsk, Belarus).

The welcome addresses were sent to the organizers and participants of the conference by First Deputy Chair of the Committee on Science and Education of the Verkhovna Rada of Ukraine Oleksandr Spivakovsky and Director of the Section of Mathematical, Physical, and Technical Sciences of the Shevchenko Scientific Society in America Roman Brukh.

The working languages of the conference were English and Ukrainian. Most of the lectures were delivered in English.

On September 26, participants of the conference had a trip to Zhuravka in Chernihiv region, the native village of Voronoï. They held a memorial service on the grave of Georgy and Theodosiy Voronoïs and visited the school local lore museum.

November 2018

Professor Mykola Pratsiovytyi
Editor of the volume,
Chair of the Organizing Committee

Вітання від Верховної Ради

21 вересня 2018 року

Від імені Комітету з питань науки і освіти, а також від себе особисто вітаю учасників Шостої міжнародної конференції з аналітичної теорії чисел і просторових мозаїк, присвяченої 150-річному ювілею вченого-математика світового рівня Георгія Феодосійовича Вороного, українця з походження!

Георгій Вороний займає одне з почесних місць в історії математичної науки в світі, саме тому його ім'я увійшло до ювілейного календаря, затвердженого Постановою Верховної Ради України «Про відзначення пам'ятних дат і ювілеїв у 2018 році» № 2287-VIII від 8 лютого 2018 р.

Працюючи в області теорії чисел, Георгій Вороний своїми науковими працями заснував і розвинув основні напрямки досліджень в цій царині науки: аналітичну теорію чисел, алгебраїчну теорію чисел, став одним із засновників нової галузі знань — геометрії чисел. «Діаграми Вороного» — об'єкти, названі його ім'ям, дали поштовх для численних застосувань в різноманітних напрямках науки і практики: від молекулярної біології до космосу, у комп'ютерній графіці, у проблемах розпізнавання образів, штучного інтелекту тощо.

Той факт, що міжнародна конференція, присвячена розвитку наукових досягнень Вороного, проходить у Києві (а це вже шоста київська конференція) та поєднується з подорожжю до малої батьківщини Георгія Вороного і місця його поховання — села Журавка на Чернігівщині, додає вагомості вкладу українських вчених до світової наукової скарбниці, піднімає науковий рейтинг нашої країни у світі, що є важливим моментом у час становлення державності.

Коли ми говоримо про великих вчених, яких вже немає з нами, ми звертаємо особливу увагу на результати їхніх наукових розробок в довгостроковій перспективі. Пройшло вже 110 років з того часу, як професор Вороний помер у віці 40 років, але великі вчені і їхні наукові досягнення залишаються в колективній пам'яті, з цієї причини його ім'я буде тривати вічно в історії математики і природничих наук в цілому.

Маю надію, що наукова конференція, яка розпочинається тут, у Києві, 24 вересня, буде успішною і приверне пильну увагу не тільки з боку самих учасників зібрання, але і широкої громадськості.

З повагою

народний депутат Олександр Співаковський,
перший заступник голови Комітету Верховної Ради України
з питань науки і освіти

Welcome Address by the Parliament

September 21, 2018

On behalf of the Committee on Science and Education and on my own behalf, let me welcome the participants of the Sixth International Conference on Analytic Number Theory and Spatial Tessellations dedicated to the 150th anniversary of the prominent mathematician, Georgy Feodosiyovych Voronoï, a Ukrainian by origin!

G. Voronoï occupies one of the notable places in the history of mathematics in the world, and his name is in the calendar of celebrations approved by the Resolution of the Verkhovna Rada of Ukraine "On Celebration of Memorial Dates and Anniversaries in 2018" No. 2287-VIII of February 8, 2018.

Georgy Voronoï worked in the field of number theory. In his scientific works, he founded and developed the main directions of research in this field of science: the analytic number theory, the algebraic number theory, and became one of the founders of a new field of knowledge – the geometry of numbers. "Voronoi diagrams" are the objects named after him. They gave impetus to numerous applications in a variety of fields of science and practice, from molecular biology to cosmos, in computer graphics, in problems of pattern recognition, artificial intelligence, and so on.

This international conference devoted to the development of scientific achievements of Voronoï is taking place in Kyiv (this is the sixth Kyiv conference) and is combined with a trip to the small homeland of Georgy Voronoï and his burial place (it is the village Zhuravka in Chernihiv region). This fact adds value to the contribution of Ukrainian scholars to the world scientific treasure, raises scientific rating of our country in the world, which is an important moment for our state formation.

When we talk about the great scientists that are not already with us, we pay special attention to the results of their scientific developments in the long-term perspective. 110 years have passed since Professor Voronoï died at the age of 40, but great scholars and their scientific achievements remain in collective memory. For this reason, his name will last forever in the history of mathematics and the natural sciences as a whole.

I hope that the scientific conference opening here, in Kyiv, on September 24, will be successful and will attract attention not only of the participants of the meeting itself, but also of the wide public.

Sincerely,

Member of Parliament Oleksandr Spivakovskyi
First Deputy Chair of the Committee on Science and Education
of the Verkhovna Rada of Ukraine

Вітання від Наукового товариства ім. Шевченка в Америці (НТШ-А)

22 вересня 2018 року

Наукове товариство імені Шевченка в Америці, а особливо члени Математично-фізично-технічної секції НТШ-А, складають щирий привіт організаторам та учасникам Шостої міжнародної конференції з аналітичної теорії чисел та просторових мозаїк, що проходитиме 24–28 вересня 2018 року в Києві. Ми дуже раді, що ця конференція присвячена 150-річчю відомого українського математика Георгія Вороного.

Праці Г. Вороного відрізняються ретельністю та винятковою глибиною викладу. Майже кожна з них послужила основою і натхненням для розвитку нових напрямків наукових досліджень, що призводять до нових досягнень.

Ім'я Георгія Вороного відоме фахівцям усього світу, а його наукові роботи широко використовуються в багатьох галузях науки і прикладних дослідженнях. Крім того, ми пишаємося його українським походженням і хочемо поділитися цим з міжнародною спільнотою.

Ми віримо, що ця конференція зробить значний внесок у розвиток математики, посилення зв'язків у міжнародній математичній спільноті та поширення знань про Георгія Вороного як видатного математика, якого породила та дала світові Українська Земля.

Від імені Наукового товариства ім. Шевченка в Америці бажаю великих успіхів у роботі конференції.

Зі щирою повагою

д-р Роман Брух,
директор Секції математичних, фізичних та технічних наук
НТШ-А

Welcome Address by the Shevchenko Scientific Society in America (NTSh-A)

September 22, 2018

The Shevchenko Scientific Society in America and the members of the Mathematical, Physical, and Technical Section of the NTSh-A wish to extend a hearty congratulations to the organizers and participants of the Sixth International Conference on Analytic Number Theory and Spatial Tessellations, which will take place on September 24–28, 2018 in Kyiv. We are delighted that the conference is dedicated to the 150th anniversary of the renowned Ukrainian mathematician, Georgy Voronoï.

The works of Georgy Voronoï are revered because of their complexity and their tremendous scope. His work has also served as a foundation and inspiration for the development of new directions in scientific research leading to new advances.

Georgy Voronoï's reputation is known to professionals around the world, and his scientific works are widely used in many fields of science and applied research. Additionally, we are proud of his Ukrainian heritage, which we want to share with the international community.

We believe that this conference will make significant contributions to the development of mathematics, the strengthening of connections in the international mathematical community, and the dissemination of knowledge about Georgy Voronoï as an outstanding Ukrainian mathematician.

On behalf of the members of the Shevchenko Scientific Society in America, I wish the conference great success.

My sincerest regards,

Dr. Roman Brukh
Director of the Section of Mathematical, Physical, and Technical Sciences of NTSh-A

Conference Organization Structure

International Program Committee

Section of Pure Mathematics

Jean-Daniel Boissonnat	INRIA, France
Liliya Boytsun	Oles Honchar Dnipro National University, Ukraine
Johannes Buchmann	Technische Universität Darmstadt, Germany
Karma Dajani	Utrecht University, the Netherlands
Yuriy Drozd	Institute of Mathematics NAS of Ukraine, Ukraine
Driss Essouabri	Université Jean Monnet, France
Aleksandar Ivić	University of Belgrade, Serbia
Matti Jutila	University of Turku, Finland
Shigeru Kanemitsu	Kindai University, Japan
Khrystyna Kuchminska	Pidstryhach Institute for Applied Problems in Mechanics and Mathematics NAS of Ukraine, Ukraine
Vichian Laohakosol	Kasetsart University, Thailand
Antanas Laurinčikas	Vilnius University, Lithuania
Kohji Matsumoto	Nagoya University, Japan
Štefan Porubský	Institute of Computer Science of Academy of Sciences of the Czech Republic, Czech Republic
Mykola Pratsiovytyi	National Pedagogical Dragomanov University, Ukraine
Andrzej Schinzel	Institute of Mathematics PAN, Poland
Jörn Steuding	University of Würzburg, Germany
Jörg Thuswaldner	Montanuniversität Leoben, Austria
Robert Tichy	Technische Universität Graz, Austria
Grygoriy Torbin	National Pedagogical Dragomanov University, Ukraine
László Tóth	University of Pécs, Hungary
Pavel Varbanets	Mechnikov National University, Odesa, Ukraine

Section of Voronoi Diagrams

Franz Aurenhammer	Technische Universität Graz, Austria
Sergiy Butenko	Texas A&M University, USA
Sung Nok Chiu	Hong Kong Baptist University, Hong Kong
Christopher Gold	University of Glamorgan, UK

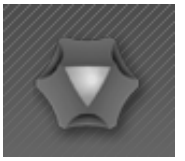
Bert Jüttler	Johannes Kepler Universität Linz, Austria
Deok-Soo Kim	Voronoi Diagram Research Center, Hanyang University, Korea
Igor Kovalenko	Institute of Cybernetics NAS of Ukraine, Ukraine
Ihor Mryglod	Institute for Condensed Matter Physics NAS of Ukraine, Ukraine
Atsuyuki Okabe	University of Tokyo, Japan
Evanthia Papadopoulou	Università della Svizzera italiana, Switzerland
Marjorie Senechal	Smith College, USA
Masaharu Tanemura	Institute of Statistical Mathematics, Tokyo, Japan
Vasyl Tereshchenko	Taras Shevchenko National University of Kyiv, Ukraine
Rien van de Weygaert	University of Groningen, the Netherlands
Yaroslav Yatskiv	Main Astronomical Observatory NAS of Ukraine, Ukraine
Ihor Yukhnovskii	Institute for Condensed Matter Physics NAS of Ukraine, Ukraine

Organizing Committee

Mykola Pratsiovytyi, Chair	National Pedagogical Dragomanov University
Halyna Syta, Secretary	National Pedagogical Dragomanov University
Oleksandr Baranovskyi	Institute of Mathematics NAS of Ukraine
Olga Bondarenko	National Pedagogical Dragomanov University
Yanina Goncharenko	National Pedagogical Dragomanov University
Tetiana Gulak	National Pedagogical Dragomanov University
Yevheniia Kalashnikova	National Pedagogical Dragomanov University
Vita Markitan	National Pedagogical Dragomanov University
Roman Nikiforov	National Pedagogical Dragomanov University
Roman Osaulenko	Institute of Mathematics NAS of Ukraine
Sofiia Ratushniak	Institute of Mathematics NAS of Ukraine
Ihor Savchenko	National Pedagogical Dragomanov University
Grygoriy Torbin	National Pedagogical Dragomanov University
Svitlana Vaskevych	Institute of Mathematics NAS of Ukraine
Natalya Vasylenko	Institute of Mathematics NAS of Ukraine
Irina Yehorchenko	Institute of Mathematics NAS of Ukraine

Conference Venue. The Sixth International Conference on Analytic Number Theory and Spatial Tessellations took place in Kyiv (Ukraine) on September 24–28. The sessions of the conference were held at the National Pedagogical Dragomanov University (9 Pyrohova St., Kyiv). On September 26, the participants of the conference had a small trip to the village Zhuravka, the place where Georgy Voronoï was born and where according to his last will, he was buried.

Our Sponsors and Supporters.



Voronoi Diagram Research Center, Hanyang University, Seoul, Korea



Ministry of Education and Science of Ukraine

The Organizing Committee of the Conference expresses the sincere gratitude to our sponsors. The organization of the conference would not have been possible without their funding and support.

We also would like to express the deep gratitude to Professor Jörn Steuding (University of Würzburg, Germany) and to Professor Deok-Soo Kim (Voronoi Diagram Research Center, Hanyang University, Seoul, Korea) for the large work during the conference preparation and to all the experts who reviewed the papers of the book.

Organizing Committee of the Conference

* * * * * * * * * * * * * * *

Note, that a reader can find here two spellings of the G. Voronoï's name: Voronoï and Voronoi. This established historically.

G. Voronoï himself used his name in such writing: Voronoï, so his followers in Pure Mathematics usually use his name in the author's spelling. But the term "Voronoi diagram" is used with the letter "i" at the end, and the experts in Voronoi diagrams applications hold on this spelling.

Voronoï's Impact on Modern Science, Book 6, Volume 1
Proceedings of the Sixth International Conference on Analytic Number Theory and Spatial Tessellations

TWO ARITHMETIC PROPERTIES OF REAL EXPONENTIAL POLYNOMIALS

Vichian Laohakosol and Pattira Ruengsinsub (*Thailand*)

Abstract. The following two arithmetic properties, which hold for polynomials, are investigated for real exponential polynomials.

(1) If $A(x)$ and $B(x)$ are two real exponential polynomials taking integer values for each positive integer s and if $B(s) \mid A(s)$, is it true that $A(x)/B(x)$ is a real exponential polynomial?
(2) If $A(x)$ is a real exponential polynomial such that $A(s)^{1/m}$ is integral for each positive integer s, is it true that $A(x)^{1/m}$ is a real exponential polynomial?

It is shown that both properties continue to hold subject to certain restrictions.

Introduction

The following two classical arithmetic properties of polynomials are well known.

1) The quotient of two polynomials $A(x)$ and $B(x)$ with rational coefficients is again a polynomial whenever $A(n)/B(n) \in \mathbb{Z}$ for an infinite set of integers n.
2) If $f(x) \in \mathbb{Z}[x]$ is an m^{th} power ($m \in \mathbb{N}$) for every $x \in \mathbb{N}$, then $f(x) = (g(x))^m$ identically, where $g(x) \in \mathbb{Z}[x]$ ([9, Part VIII, Chapter 2]).

As mentioned in [5], the first property is easily proved using the division algorithm. The second property has been extended over the years in various directions, see, e.g., [2, 3, 4, 10]. Here, we investigate the validity of these two properties when polynomials are replaced by real exponential polynomials. There have already appeared some results in this setting, e.g., Lovász in [6] established the two properties for the so-called finite Dirichlet series, which is an exponential sum with integer coefficients and integer exponents. Perelli and Zannier in [7, 8] established both properties for exponential polynomials with integral polynomial coefficients

Received July 24, 2018; revised October 29, 2018.
2010 *Mathematics Subject Classification.* Primary 33B10; Secondary 11B37.
Key words and phrases. Exponential polynomial, arithmetic properties.

and integer exponents. Lewis and Morton in [5] proved the first property for rational and algebraic number fields, subject to certain conditions, for polynomials in several variables. Rumely and van der Poorten in [12] proved the second property (also known as Pisot's conjecture) in the case of number field subject to certain restriction.

Throughout this work, an *exponential polynomial* is an expression of the form

$$A(x) = \sum_{k=1}^{K} a_k(x) e^{\alpha_k x}, \tag{1}$$

where $a_k(x) \in \mathbb{C}[x] \setminus \{0\}$ and α_k $(k = 1, \ldots, K)$ are distinct complex numbers. The $a_k(x)$'s and the α_k's are referred to as the *coefficients* and the *frequencies*, respectively.

Whenever we write an exponential polynomial (1)*, the frequencies will always be so arranged with non-decreasing real parts, i.e.,*

$$\Re\alpha_1 \leq \Re\alpha_2 \leq \cdots \leq \Re\alpha_K. \tag{2}$$

The frequency α_K is called its *largest* frequency. If all $a_k(x) \in \mathbb{R}[x] \setminus \{0\}$ and all $\alpha_k \in \mathbb{R}$, such an exponential polynomial is termed a *real exponential polynomial*.

In the next section, we show that, subject to certain restrictions, the two above-mentioned properties continue to hold for real exponential polynomials. In the third and fourth sections, we consider real exponential polynomials with *positive rational exponent bases* (RB-EP), i.e., all $e^{\alpha_k} \in \mathbb{Q}$, and *rational exponential polynomials* (REP), which are RB-EP's whose polynomials have rational coefficients, i.e., all $a_k(x) \in \mathbb{Q}[x]$. For the latter exponential polynomials, the restrictions imposed for general real exponential polynomials can be removed and stronger results can be obtained.

In our work here, we use a combination of three methods. The first, for general real exponential polynomials, is due to Lewis and Morton in [5] which makes crucial use of a result of Pisot [1, Theorem IV, p. 138] (see (D) below). The second, for real exponential polynomials with rational exponent bases is due to Lovász in [6] which yields stronger results due to the fact that such an exponential polynomial satisfies a linear recurrence relation with integral coefficients, and the third which involves certain important reductions is due to Perelli–Zannier, [8]; these reductions were further extended and made precise in [12].

Let us first collect certain well-known results concerning rational functions, linear recurrence relations and exponential polynomials; for general references, see [13, Part C, pp. 32–39] and [1, Chapter VIII].

(**A**) ([5, Section 2]) A formal power series

$$\sum_{n\geq 0} a_n x^n \in \mathbb{C}[[x]]$$

represents a rational function if and only if the sequence $\{a_n\}$ is a linear recurrence sequence, i.e., if and only if there are complex numbers $c_0(\neq 0)$, c_1, ..., $c_r(\neq 0)$ for which

$$c_r a_{n+r} + c_{r-1} a_{n+r-1} + \cdots + c_1 a_{n+1} + c_0 a_n = 0 \quad (n \geq n_0), \tag{3}$$

where n_0 is some sufficiently large integer.

(**B**) ([5, Section 2]) If α_1, ..., α_s are the distinct roots with respective multiplicities e_1, ..., e_s of the polynomial

$$\mathcal{C}(X) := c_r X^r + c_{r-1} X^{r-1} + \cdots + c_1 X + c_0 = c_r \prod_{j=1}^{s} (X - \alpha_j)^{e_j}, \tag{4}$$

where the c_k's are as in (3), then the elements

$$n^j \alpha_k^n, \quad 1 \leq k \leq s, \quad 0 \leq j \leq e_k - 1,$$

are linearly independent over $\mathbb{C}$, and form a basis for the solution space of (3). Any solution $\{a_n\}$ of (3) has a unique representation of the form

$$a_n = \sum_{k=1}^{s} p_k(n) \alpha_k^n \quad (n \geq n_0), \tag{5}$$

where $p_k(x) \in \mathbb{C}[x]$, and $\deg p_k(x) \leq e_k - 1$. Conversely, any sequence $\{a_n\}$ of exponential polynomial values defined by (5) satisfies a recurrence of the form (3).

In terms of the operator E defined by $Ef(n) := f(n+1)$, since

$$\mathcal{C}(E) := c_r E^r + c_{r-1} E^{r-1} + \cdots + c_1 E + c_0,$$

the recurrence (3) takes the form $\mathcal{C}(E)(a_n) = 0$ and the above result shows that $\mathcal{C}(E)$ annihilates all terms $n^j \alpha_k^n$ $(1 \leq k \leq s, 0 \leq j \leq e_k - 1)$.

(**C**) ([1, p. 138]) Let $\{x_n\}$ be a sequence of real numbers satisfying a linear recurrence of the form (3), and let $\{A_n\}$ be a sequence of integers. If

$$\sum_{n=0}^{\infty} |x_n - A_n|^2 < \infty,$$

then the sequence $\{A_n\}$ also satisfies a linear recurrence of the form (3).

(**D**) ([1, p. 137]) Let $\{x_n\}$ be a sequence of numbers satisfying a linear recurrence of the form (3). If all the x_n are rational, then c_0, c_1, ..., c_r can be taken without loss of generality to be rational.

(**E**) ([14, Example 2, p. 422]) Let

$$F(z) = \sum_{k=1}^{m} P_k(z) e^{Q_k(z)},$$

where P_k, $Q_k \in \mathbb{C}[z]$ $(k = 1, \ldots, m)$. If all the polynomials $Q_k(x)$ $(k = 1, \ldots, m)$ have real coefficients, then the number of zeros of F on the real line is bounded.

Next, we list those earlier related results known to us.

A. A.1) ([6, Theorem 1]) Assume that

$$f(s) = \sum_{i=1}^{n} a_i i^s, \quad g(s) = \sum_{i=1}^{m} b_i i^s \quad (a_i, b_i \in \mathbb{Z};\ a_n b_m \neq 0)$$

are finite Dirichlet series such that $f(s) \mid g(s)$ for every $s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Then $n \mid m$ and $h(s) = \left(\frac{m}{n} - 1\right)! \frac{g(s)}{f(s)}$ is a finite Dirichlet series.

A.2) ([6, Theorem 2]) Let

$$f(s) = \sum_{i=1}^{n} a_i i^s \quad (a_i \in \mathbb{Z})$$

be a finite Dirichlet series and assume that $f(s)$ is the k^{th} power of an integer for every $s \in \mathbb{N}_0$. Then $f(s)$ is the k^{th} power of a finite Dirichlet series.

B. B.1) ([5, Lemma 3]) Let

$$a_n = \sum_{i=1}^{r} c_i \alpha_i^n, \quad b_n = \sum_{j=1}^{s} d_j \beta_j^n,$$

where the c_i and $d_j \in \mathbb{R} \backslash \{0\}$, and where the α_i and β_j are positive and respectively pairwise distinct. If $a_n/b_n \in \mathbb{Z}$ for those n for which $b_n \neq 0$, then

$$a_n = b_n \sum_{k=1}^{t} u_k \gamma_k^n \quad (n \in \mathbb{N}_0),$$

where the u_k and γ_k are real and $\gamma_k > 0$. Moreover, γ_k lies in the multiplicative group G generated by the α_i and β_j.

B.2) ([5, Lemma 4]) Let

$$a_n = \sum_{i=1}^{r} c_i q_i^n \quad (n \geq n_0),$$

where $c_i, q_i \in \mathbb{Q}$, $c_i q_i \neq 0$ and the q_i are distinct. If $a_n \in \mathbb{Z}$ for $n \geq n_0$, then $q_i \in \mathbb{Z}$ for $1 \leq i \leq r$.

C. C.1) ([8, Theorem 1]) Let A be the integral domain of the functions $f : \mathbb{N} \to \mathbb{Z}$ of the form

$$f(n) = \sum_{j=1}^{M} P_j(n) a_j^n,$$

where $P_j(n) \in \mathbb{Z}[n]$, $a_j \in \mathbb{N}$. If $g(n) = f_1(n)/f_2(n)$, where f_1 and $f_2 \in A$, and if $g(m) \in \mathbb{Z}$ for every $m \in \mathbb{N}$, $m > m_0$, then $Cg \in A$ for some integer C.

C.2) ([8, Theorem 2]) Let B be the ring of the functions

$$g : \mathbb{N} \to \mathbb{R}$$

of the form

$$g(m) = \sum_{i=1}^{N} C_i \left(f_i(m)\right)^{1/k_i}, \quad C_i \in \mathbb{Z},\ k_i \in \mathbb{N},\ f_i \in A,\ f_i(m) \geq 0$$

for every $m \in \mathbb{N}$. If $g(m) \in \mathbb{Z}$ for every m belonging to a fixed sequence of natural numbers which intersects every arithmetic progression, then $Cg \in A$ for some integer C.

1. Real exponential polynomials

We start by deriving a result which is a division algorithm for exponential polynomials, extending, yet using essentially the same proof, the one due originally to Ritt in [11].

Lemma 1 ([11])**.** *Let*

$$A(x) = \sum_{k=1}^{K} a_k(x) e^{\alpha_k x}, \quad B(x) = \sum_{j=1}^{J} b_j(x) e^{\beta_j x}$$

be two exponential polynomials as defined through (1) *and* (2)*. If* $\Re\beta_J > \Re\beta_{J-1}$*, then there are* (*quotient*) *exponential polynomial* $Q(x)$ *whose largest frequency has positive real part* (*provided* $Q(x) \not\equiv 0$)*,* (*remainder*) *exponential polynomial*

$$R(x) = r_1(x) e^{\gamma_1 x} + \cdots + r_\ell(x) e^{\gamma_\ell x},$$

and $N \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ *satisfying*

$$b_J(x)^N A(x) = Q(x) B(x) + R(x), \tag{6}$$

with either $R(x) \equiv 0$ *or* $\Re\beta_J > \Re\gamma_\ell$*.*

Proof. If $\Re\beta_J > \Re\alpha_K$, then (6) holds with $Q(x) = 0$, $R(x) = A(x)$, $N = 0$. Assume then that

$$\Re\alpha_K \geq \Re\alpha_{K-1} \geq \cdots \geq \Re\alpha_{K-j} \geq \Re\beta_J > \Re\alpha_{K-j-1} \geq \cdots \geq \Re\alpha_1.$$

Consider the exponential polynomial

$$C(x) = A(x) - \frac{a_K(x)e^{\alpha_K x} + \cdots + a_{K-j}(x)e^{\alpha_{K-j}x}}{b_J(x)e^{\beta_J x}} \cdot B(x). \tag{7}$$

If $B(x)$ consists only of one term, then $C(x)$ is either 0 or its largest frequency has real part smaller than $\Re\beta_J$, so that in (6), we have $N = 1$, and

$$\begin{aligned} Q(x) &= a_K(x)e^{(\alpha_K-\beta_J)x} + \cdots + a_{K-j}(x)e^{(\alpha_{K-j}-\beta_J)x}, \\ R(x) &= b_J(x)\left\{a_1(x)e^{\alpha_1 x} + \cdots + a_{K-j-1}(x)e^{\alpha_{K-j-1}x}\right\}. \end{aligned}$$

Suppose now that $B(x)$ has at least two terms. If $C(x) \equiv 0$, then the result is trivial. If $C(x)$ is nonzero, then its highest term (i.e., term with largest frequency) is of the form

$$\frac{a_K(x)b_{J-1}(x)}{b_J(x)}\, e^{(\alpha_K-\beta_J+\beta_{J-1})x} \quad \text{or} \quad a_{K-j-1}(x)e^{\alpha_{K-j-1}x}.$$

If $\Re(\alpha_K - \beta_J + \beta_{J-1}) < \Re\beta_J$, multiplying throughout by $b_J(x)$, we have in (7) a representation (6). Otherwise, we repeat the process just described, subjecting $C(x)$ to the treatment received by $A(x)$ to get an expression of the form

$$C_2(x) = C(x) - \frac{\frac{a_K(x)b_{J-1}(x)}{b_J(x)}\, e^{(\alpha_K-\beta_J+\beta_{J-1})x} + \cdots}{b_J(x)e^{\beta_J x}} \cdot B(x).$$

If $C_2(x)$ is nonzero, then the real part of its largest frequency (if not $< \Re\beta_J$) must be equal to $\Re(\alpha_K - 2(\beta_J - \beta_{J-1}))$. Since $\Re(\beta_J - \beta_{J-1})$ is a fixed quantity, we arrive in a finite number of steps an expression whose largest frequency has real part $< \Re\beta_J$ and after clearing the denominators, which is a power of $b_J(x)$, we obtain at a representation (6). □

Remarks. 1) With a little more work, it is not difficult to show that the exponential polynomials $Q(x)$, $R(x)$ and the integer N are uniquely determined.
2) In Lemma 1, if both $A(x)$ and $B(x)$ are real exponential polynomials, then both $Q(x)$ and $R(x)$ are also real exponential polynomials. Moreover, if all exponential terms e^{α_k}, e^{β_j} are rational, then so are all the exponential terms in $Q(x)$ and $R(x)$.

1.1. **Quotient.**

Theorem 1. *Let*

$$A(x) = \sum_{k=1}^{K} a_k(x) e^{\alpha_k x}, \quad B(x) = \sum_{j=1}^{J} b_j(x) e^{\beta_j x}$$

be two real exponential polynomials with

$$\beta_J > \beta_{J-1} > \cdots > \beta_1.$$

Assume that for every $s \in \mathbb{N}$, the numbers $b_J(s)$, $A(s)$ and $B(s)$ $(\neq 0)$ are integers such that $B(s) \mid A(s)$. Then there is a positive integer N such that $b_J(x)^N A(x)/B(x)$ is itself a real exponential polynomial.

Proof. Using Lemma 1 and taking $s \in \mathbb{N}$, we have

$$b_J(s)^N \frac{A(s)}{B(s)} = Q(s) + \frac{R(s)}{B(s)}.$$

Letting $R(x) = r_1(x)e^{\gamma_1 x} + \cdots + r_\ell(x)e^{\gamma_\ell x}$, since $\gamma_\ell = \Re\gamma_\ell < \Re\beta_J = \beta_J$, for s sufficiently large, say $s \geq s_0$, we see that

$$\left| \frac{R(s)}{B(s)} \right| \leq c_1 e^{c_2(\gamma_\ell - \beta_J)s},$$

for some positive constants c_1, c_2. Thus,

$$\sum_{s=0}^{\infty} \left| b_J(s)^N \frac{A(s)}{B(s)} - Q(s) \right|^2 < \infty.$$

Since $Q(x)$ is a real exponential polynomial, by the results (**A**) and (**B**), the real sequence $\{Q(s)\}_{s\in\mathbb{N}}$ satisfies a linear recurrence relation. By the results (**C**) and (**D**), the sequence of integers $\{b_J(s)^N A(s)/B(s)\}_{s\geq s_0}$ also satisfies a linear recurrence relation. Thus, by the result (**B**) again, we have

$$b_J(s)^N \frac{A(s)}{B(s)} = \sum_{m=1}^{M} f_m(s) e^{\eta_m s} \quad (s \geq s_0),$$

where $f_m(s) \in \mathbb{C}[s] \setminus \{0\}$, and η_m $(m = 1, \ldots, M)$ are distinct complex numbers. This yields

$$b_J(s)^N \sum_{k=1}^{K} a_k(s) e^{\alpha_k s} = \left(\sum_{j=1}^{J} b_j(s) e^{\beta_j s} \right) \left(\sum_{m=1}^{M} f_m(s) e^{\eta_m s} \right) \quad (s \geq s_0). \quad (8)$$

We claim that each e^{η_m} lies in the multiplicative group generated by the e^{α_k}, e^{β_j}, which in turn implies that each η_m is a real number. Suppose not. Let $e^{\eta_{m_0}}$ be the e^{η_m} of least absolute value and smallest argument which is not in the said multiplicative group. Since $\beta_1 < \beta_j$ $(j = 2, \ldots, J)$,

the term $b_1(s)f_{m_0}(s)\left(e^{\beta_1 s}e^{\eta_{m_0}s}\right)$ in the product on the right-hand side does not combine with any other term in the product. Hence, when s is sufficiently large, this term must equal some term $b_J(s)^N a_k(s)e^{\alpha_k s}$ on the left-hand side. This is possible only when $e^{\beta_1}e^{\eta_{m_0}} = e^{\alpha_k}$, which is a contradiction, and the claim is proved.

It also follows from (8) that each polynomial $f_m(x)$ has real coefficients, i.e., $\sum f_m(x)e^{\eta_m x}$ is a real exponential polynomial. Now, by (**E**), the real exponential polynomials $b_J(x)^N A(x) - B(x)\sum_{m=1}^{M} f_m(x)e^{\eta_m x}$ can only have a finite number of real zeros, which then shows that it vanishes identically, and the theorem follows. □

Remarks. 1) If $b_J(x)$ is a nonzero real constant, with only a slight modification of the above proof, Theorem 1 still holds without the condition $b_J(s) \in \mathbb{Z}$ for all $s \in \mathbb{N}$.
2) As is well known and is easily checked, the condition $b_J(s) \in \mathbb{Z}$ $(s \in \mathbb{N})$ implies that $b_J(x) \in \mathbb{Q}[x]$.

1.2. **Powers.**

Theorem 2. *Let $m \in \mathbb{N}$, $m \geq 2$, and let*

$$A(x) = \sum_{k=1}^{K-1} a_k(x)e^{\alpha_k x} + a_K e^{\alpha_K x}, \quad a_K > 0, \quad \alpha_1 < \alpha_2 < \cdots < \alpha_K,$$

be a real exponential polynomial. Assume that $A(s)$ is an m^{th} power of an integer for sufficiently large $s \in \mathbb{N}$. Then $A(x)$ is an m^{th} power of another real exponential polynomial.

Proof. Let

$$U(x) := \frac{a_{K-1}(x)}{a_K}\, e^{(\alpha_{K-1}-\alpha_K)x} + \cdots + \frac{a_1(x)}{a_K}\, e^{(\alpha_1-\alpha_K)x},$$

so that

$$A(x)^{1/m} = \left(a_K e^{\alpha_K x}\right)^{1/m} \left\{1 + U(x)\right\}^{1/m},$$

where $a_K^{1/m}$ denotes one of its real m-th roots. For sufficiently large $s \in \mathbb{N}$, say $s \geq s_0$, we have

$$\begin{aligned} |U(s)| &\leq \left|\frac{a_{K-1}(s)}{a_K}\right| e^{(\alpha_{K-1}-\alpha_K)s} + \cdots + \left|\frac{a_1(s)}{a_K}\right| e^{(\alpha_1-\alpha_K)s} \\ &\leq c_1 s^{N_0} e^{(\alpha_{K-1}-\alpha_K)s} < e^{(\alpha_{K-1}-\alpha_K)s/m} < 1, \end{aligned} \tag{9}$$

where $c_1 > 0$ and $N_0 \in \mathbb{Z}$ are constants, and so we can write

$$(1 + U(s))^{1/m} = \sum_{j=0}^{\infty} \binom{1/m}{j} U(s)^j.$$

For $t \in \mathbb{N}$, let

$$R_t(s) := (a_K e^{\alpha_K s})^{1/m} \sum_{j=t+1}^{\infty} \binom{1/m}{j} U(s)^j.$$

Using $\left|\binom{1/m}{j}\right| \le 1$ and (9), for $s \ge s_0$, we have

$$\begin{aligned} |R_t(s)| &\le \left|(a_K e^{\alpha_K s})^{1/m}\right| \sum_{j=t+1}^{\infty} \left|U(s)^j\right| \\ &\le \left|(a_K e^{\alpha_K s})^{1/m}\right| \frac{e^{(\alpha_{K-1}-\alpha_K)s(t+1)/m}}{1-e^{(\alpha_{K-1}-\alpha_K)s/m}}. \end{aligned}$$

Choosing a fixed t sufficiently large, we see that

$$\begin{gathered} |R_t(s)| \to 0 \quad (s \to \infty), \\ \sum_{s \ge s_0} |R_t(s)| < \infty, \\ \sum_{s \ge s_0} |R_t(s)|^2 < \infty. \end{gathered}$$

Next, write

$$\begin{aligned} &(a_K e^{\alpha_K s})^{1/m} \sum_{j=0}^{t} \binom{1/m}{j} U(s)^j \\ &\qquad = \sum_{i=1}^{N} u_i(s)\big(e^{\alpha_K/m} e^{\gamma_i}\big)^s + \sum_{\ell=1}^{N'} v_\ell(s)\big(e^{\alpha_K/m} e^{\delta_\ell}\big)^s, \end{aligned}$$

for some $N, N' \in \mathbb{N}$; $u_i(s), v_\ell(s) \in \mathbb{R}[s]$; and γ_i, δ_ℓ are real numbers with

$$e^{\alpha_K/m} e^{\gamma_i} \ge 1, \quad e^{\alpha_K/m} e^{\delta_\ell} < 1 \quad (i = 1, \dots, N; \quad \ell = 1, \dots, N').$$

For brevity, put

$$H(s) = \sum_{i=1}^{N} u_i(s)\big(e^{\alpha_K/m} e^{\gamma_i}\big)^s, \quad R_2(s) = \sum_{\ell=1}^{N'} v_\ell(s)\big(e^{\alpha_m/k} e^{\delta_\ell}\big)^s,$$

so that

$$A(s)^{1/m} = H(s) + R_2(s) + R_t(s).$$

Clearly,

$$\begin{gathered} |R_2(s)| \to 0 \quad (s \to \infty), \\ \sum_{s \ge s_0} |R_2(s)| < \infty, \end{gathered}$$

$$\sum_{s\geq s_0}|R_2(s)|^2<\infty,$$
$$\sum_{s\geq s_0}|R_t(s)R_2(s)|<\infty.$$

Letting

$$R(s)=R_t(s)+R_2(s),$$

we see that

$$\sum_{s=1}^{\infty}|R(s)|^2\leq\sum_{s=1}^{\infty}\left\{|R_t(s)|^2+|R_2(s)|^2+2\,|R_t(s)R_2(s)|\right\}<\infty.$$

Since $H(s)$ is a real exponential polynomial in s, by (**B**), the real sequence $\{H(s)\}_{s\geq s_0}$ satisfies a linear recurrence relation and since $\left\{A(s)^{1/m}\right\}$ is a sequence of integers, using

$$\sum_{s=1}^{\infty}\left|A(s)^{1/m}-H(s)\right|^2=\sum_{s=1}^{\infty}|R_2(s)+R_t(s)|^2<\infty,$$

we deduce from (**C**) that the sequence $\left\{A(s)^{1/m}\right\}_{s\geq s_0}$ also satisfies a linear recurrence relation, and so by (**A**) and (**B**), it must be of the form

$$A(s)^{1/m}=d_1(s)e^{\eta_1 s}+\cdots+d_w(s)e^{\eta_w s},\tag{10}$$

where $d_j(s)\in\mathbb{C}[s]\setminus\{0\}$, $\eta_j\in\mathbb{C}$, i.e.,

$$A(s)=\left(d_1(s)e^{\eta_1 s}+\cdots+d_w(s)e^{\eta_w s}\right)^m\quad(s\geq s_0).$$

We claim that each e^{η_j} is real. Suppose not; let $e^{\eta_{j_0}}$ be the e^{η_j} of least absolute value and smallest argument which is not real. Clearly, the term $(d_{j_0}(s)e^{\eta_{j_0}s})^m$ in the product on the right-hand side does not combine with any other term in the product. Thus by (**B**), this term must equal some term $a_k(s)e^{\alpha_k s}$ on the left-hand side. This is possible only when $e^{m\eta_{j_0}}=e^{\alpha_k}$, contradicting the fact that η_{j_0} is not real, and the claim is proved.

From (10), since the left-hand side is an integer for all large s, it follows that each polynomial $d_j(s)$ has real coefficients, i.e.,

$$d_1(s)e^{\eta_1 s}+\cdots+d_w(s)e^{\eta_w s}$$

is a real exponential polynomial. Finally, since the real exponential polynomial

$$A(x)-\left(d_1(x)e^{\eta_1 x}+\cdots+d_w(x)e^{\eta_w x}\right)^m$$

has infinitely many integer zeros, by (**E**), it vanishes identically, and the theorem follows. □

2. Exponential polynomials with rational exponent bases, RB-EP

In this section, we consider real exponential polynomials of the form

$$A(x) = a_1(x)A_1^x + \cdots + a_K(x)A_K^x, \tag{11}$$

where all exponent bases A_k are *positive rational numbers* arranged in an increasing order

$$0 < A_1 < A_2 < \cdots < A_K. \tag{12}$$

For convenience, we refer to such a real exponential polynomial with positive rational exponent bases as an RB-EP. In the special case where $a_k(x) \in \mathbb{Z}$, $A_k = k$ $(k = 1, \ldots, K)$, this RB-EP is called a *finite Dirichlet series*, [6]. The two main earlier results (Theorems 1 and 2) can be strengthened for RB-EP's using Lovász's proof in [6], which makes use of the fact that an RB-EP satisfies a linear recurrence relation with integral coefficients.

2.1. **Quotient.**

Theorem 3. *Let*

$$A(x) = a_1(x)A_1^x + \cdots + a_K(x)A_K^x, \quad 0 < A_1 < \cdots < A_K$$

with all $A_k \in \mathbb{Q}$, *and*

$$B(x) = b_1(x)B_1^x + \cdots + b_J(x)B_J^x, \quad 0 < B_1 < \cdots < B_J$$

with all $B_k \in \mathbb{Q}$, *be two RB-EP's. Assume that for every* $s \in \mathbb{N}$, *the numbers* $b_J(s)$, $A(s)$ *and* $B(s)$ $(\neq 0)$ *are integers. If* $B(s) \mid A(s)$, *then there is a positive integer* N *such that* $b_J(x)^N A(x)/B(x)$ *is itself an RB-EP.*

Proof. Using Lemma 1, we have

$$b_J(x)^N \frac{A(x)}{B(x)} = Q(x) + \frac{R(x)}{B(x)},$$

where $R(x) = r_1(x)\gamma_1^x + \cdots + r_\ell(x)\gamma_\ell^x$, $0 < \gamma_1 < \cdots < \gamma_\ell$, $\gamma_i \in \mathbb{Q}$. For $s \in \mathbb{N}$, since $\gamma_\ell < B_J$, we see that $R(s)/B(s) \to 0$ $(s \to \infty)$. Let

$$Q(x) = q_1(x)\delta_1^x + \cdots + q_t(x)\delta_t^x, \quad 0 < \delta_1 < \cdots < \delta_t, \quad \delta_i \in \mathbb{Q}.$$

Since the δ_k's are rational, let them be all distinct roots with corresponding multiplicity $\deg q_k + 1$ of the polynomial

$$\mathcal{C}(X) := \sum_{j=0}^{T} c_j X^j = c_T(X - \delta_1)^{\deg q_1 + 1} \cdots (X - \delta_T)^{\deg q_T + 1} \in \mathbb{Z}[X]$$

with degree $T := t + \deg q_1 + \cdots + \deg q_t$. By (**B**), the operator $\mathcal{C}(E)$ annihilates the elements $j^r\delta_i^j$ $(1 \le i \le t,\ 0 \le r \le \deg q_i)$, so that

$$\sum_{j=0}^{T} c_j \sum_{i=1}^{t} q_i(s+j)\delta_i^{s+j} = \sum_{i=1}^{t} \delta_i^s \sum_{j=0}^{T} c_j q_i(s+j)\delta_i^j = 0,$$

and so

$$\sum_{j=0}^{T} b_J(s+j)^N c_j \frac{A(s+j)}{B(s+j)} = \sum_{j=0}^{T} c_j \frac{R(s+j)}{B(s+j)}.$$

By assumption, the left-hand side is an integer, while the right-hand side tends to 0 as $s \to \infty$. This implies that both sides vanish for sufficiently large integer s, say $s \ge s_0$, i.e.,

$$\sum_{j=0}^{T} b_J(s+j)^N c_j \frac{A(s+j)}{B(s+j)} = 0 \quad (s \ge s_0).$$

By (**B**), we have

$$b_J(s)^N \frac{A(s)}{B(s)} = \sum_{i=1}^{t} d_i(s)\gamma_i^s \quad (s \ge s_0), \tag{13}$$

for some $d_i(s) \in \mathbb{C}[s]$, $\gamma_i \in \mathbb{C}$ $(i = 1, \ldots, t)$, and so

$$b_J(s)^N A(s) = \left(b_1(s)B_1^s + \cdots + b_J(s)B_J^s\right)\left(\sum_{i=1}^{t} d_i(s)\gamma_i^s\right) \quad (s \ge s_0). \tag{14}$$

We claim that each γ_i is rational. Suppose not; let γ_{i_0} be the γ_i of least absolute value and smallest argument which is not rational. Clearly, the term $b_1(s)d_{i_0}(s)\,(B_1\gamma_{i_0})$ in the product on the right-hand side does not combine with any other term in the product. Thus, this term must equal some term $b_J(s)a_k(s)A_k^s$ on the left-hand side, contradicting the fact that γ_{i_0} is not rational, and the claim is proved.

Equating coefficients in (13), we deduce that each $d_i(s) \in \mathbb{R}[s]$. Finally, by (**E**), the RB-EP $b_J(x)^N A(x) - B(x)\sum_{i=1}^{t} d_i(x)\gamma_i^x$ can only have a finite number of real zeros, which is a contradiction unless it vanishes identically, and the theorem follows. □

2.2. **Powers.**

Theorem 4. *Let $m \in \mathbb{N}$, $m \ge 2$, and let*

$$A(x) = \sum_{k=1}^{K-1} a_k(x)A_k^x + d(x)^m A_K^x, \quad 0 < A_1 < \cdots < A_K$$

with all $A_i \in \mathbb{Q}$, be an RB-EP with $d(x) \in \mathbb{Z}[x]\backslash\{0\}$. Assume that $A(s)$ is an m^{th} power of an integer for every $s \in \mathbb{N}$. Then there exists $t \in \mathbb{N}$ such that $d(x)^{(mt-1)m}A(x)$ is an m^{th} power of a real exponential polynomial.

Proof. Let

$$U(x) := \frac{a_{K-1}(x)}{d(x)^m}\left(\frac{A_{K-1}}{A_K}\right)^x + \cdots + \frac{a_1(x)}{d(x)^m}\left(\frac{A_1}{A_K}\right)^x,$$

so that

$$A(x)^{1/m} = d(x)A_K^{x/m}\left\{1+U(x)\right\}^{1/m}.$$

Since $0 < A_i/A_K < 1$ $(i = 1, \ldots, K-1)$, for sufficiently large $s \in \mathbb{N}$ exceeding the largest real root of $d(x)$, say $s \geq s_0$, we have

$$\begin{aligned}|U(s)| &\leq \left(\left|\frac{a_{K-1}(s)}{d(s)^m}\right| + \cdots + \left|\frac{a_1(s)}{d(s)^m}\right|\right)\left(\frac{A_{K-1}}{A_K}\right)^s \\ &\leq c_1 s^{N_0}\left(\frac{A_{K-1}}{A_K}\right)^s \leq \left(\frac{A_{K-1}}{A_K}\right)^{s/m} < 1,\end{aligned}$$

where $c_1 > 0$ and $N_0 \in \mathbb{Z}$ are constants, and so we can write

$$(1+U(s))^{1/m} = \sum_{j=0}^{\infty}\binom{1/m}{j}U(s)^j \quad (s \geq s_0).$$

For $t \in \mathbb{N}$, let

$$R_t(s) := d(s)A_K^{s/m}\sum_{j=t+1}^{\infty}\binom{1/m}{j}U(s)^j.$$

Using $\left|\binom{1/m}{j}\right| \leq 1$, we have

$$\begin{aligned}|R_t(s)| &\leq \left|d(s)A_K^{s/m}\right|\sum_{j=t+1}^{\infty}\left|U(s)^j\right| \\ &\leq |d(s)|\,\frac{\{(A_{K-1}/A_K)^{t+1}A_K\}^{s/m}}{1-(A_{K-1}/A_K)^{s/m}}. \qquad (15)\end{aligned}$$

Next, write

$$\begin{aligned}&d(s)A_K^{s/m}\sum_{j=0}^{t}\binom{1/m}{j}U(s)^j \\ &= d(s)\sum_{i=1}^{N}\frac{u_i(s)}{d(s)^{mt}}\left(A_K^{1/m}\gamma_i\right)^s + d(s)\sum_{\ell=1}^{N'}\frac{v_\ell(s)}{d(s)^{mt}}\left(A_K^{1/m}\delta_\ell\right)^s \\ &=: H(s) + R_2(s), \qquad (16)\end{aligned}$$

for some $N, N' \in \mathbb{N}$; $u_i(s)$, $v_\ell(s) \in \mathbb{R}[s]$; and all γ_i, δ_ℓ are positive rational numbers satisfying

$$A_K^{1/m}\gamma_i \geq 1, \quad A_K^{1/m}\delta_\ell < 1 \quad (i = 1, \ldots, N; \quad \ell = 1, \ldots, N').$$

Let

$$R(s) = R_t(s) + R_2(s),$$

so that

$$\begin{aligned} d(s)^{mt-1}A(s)^{1/m} &= d(s)^{mt-1}\left(H(s) + R(s)\right) \\ &= \sum_{i=1}^{N} u_i(s)\big(A_K^{1/m}\gamma_i\big)^s + d(s)^{mt-1}R(s). \end{aligned}$$

Choosing a suitably fixed t sufficiently large, we see from (15) and (16) that $|d(s)^{mt-1}R(s)| \to 0$ $(s \to \infty)$. Let

$$\varphi(X) = M\prod_{i=1}^{N}(X^m - A_K\gamma_i^m)^{\deg u_i+1} = \sum_{j=0}^{mD} e_j X^j,$$

where $D := N + \deg u_1 + \cdots + \deg u_N$, and $M \in \mathbb{N}$ is so chosen that the e_j's are integers. Thus,

$$\begin{aligned} &\sum_{j=0}^{mD} e_j\ d(s+j)^{mt-1}A(s+j)^{1/m} \\ &\quad = \sum_{i=1}^{N}\big(A_K^{1/m}\gamma_i\big)^s \sum_{j=0}^{mD} e_j\ u_i(s+j)\big(A_K^{1/m}\gamma_i\big)^j \\ &\qquad\qquad + \sum_{j=0}^{mD} e_j\ d(s+j)^{mt-1}R(s+j) \\ &\quad = \sum_{j=0}^{mD} e_j\ d(s+j)^{mt-1}R(s+j) \to 0 \quad (s \to \infty). \end{aligned}$$

Since the left-hand expression is an integer, both sides must vanish for all sufficiently large s, say $s \geq s_1$, i.e.,

$$\sum_{j=0}^{mD} e_j\ d(s+j)^{mt-1}R(s+j) = 0 = \sum_{j=0}^{mD} e_j\ d(s+j)^{mt-1}A(s+j)^{1/m} \quad (s \in \mathbb{N}, s \geq s_1). \quad (17)$$

By (**B**), this yields

$$d(s)^{mt-1}R(s) = \sum_{\nu=1}^{N}\sum_{\mu=1}^{m} c_{\nu\mu}(s)\big(A_K^{1/m}\gamma_\nu\varepsilon_\mu\big)^s \quad (s \geq s_1).$$

where $c_{\nu\mu}(s) \in \mathbb{C}[s]$, $\varepsilon_\mu = \exp(2\pi i\mu/m)$. For a given pair ν_0, μ_0, let

$$\psi(X) = \frac{\varphi(X)}{\Big(X - A_K^{1/m}\gamma_{\nu_0}\varepsilon_{\mu_0}\Big)^{\deg c_{\nu_0\mu_0}}} = \sum_{i=0}^{mD-\deg c_{\nu_0\mu_0}} f_i X^i.$$

By (**B**), we thus have

$$\begin{aligned}
&\sum_{i=0}^{mD-\deg c_{\nu_0\mu_0}} f_i\, d(s+i)^{mt-1}R(s+i) \\
&\qquad = \sum_{\nu=1}^{N}\sum_{\mu=1}^{m}\big(A_K^{1/m}\gamma_\nu\varepsilon_\mu\big)^s \sum_{i=0}^{mD-\deg c_{\nu_0\mu_0}} c_{\nu\mu}(s+i) f_i\big(A_K^{1/m}\gamma_\nu\varepsilon_\mu\big)^i \\
&\qquad = \big(A_K^{1/m}\gamma_{\nu_0}\varepsilon_{\mu_0}\big)^s c_{\nu_0\mu_0} \sum_{i=0}^{mD-\deg c_{\nu_0\mu_0}} i^{\deg c_{\nu_0\mu_0}} f_i\big(A_K^{1/m}\gamma_{\nu_0}\varepsilon_{\mu_0}\big)^i \\
&\qquad = c_{\nu_0\mu_0}\big(A_K^{1/m}\gamma_{\nu_0}\varepsilon_{\mu_0}\big)^s \left[\left(X\frac{d}{dX}\right)^{\deg c_{\nu_0\mu_0}} \psi\right]_{A_K^{1/m}\gamma_{\nu_0}\varepsilon_{\mu_0}},
\end{aligned}$$

where $c_{\nu_0\mu_0}$ denotes the leading coefficient of $c_{\nu_0\mu_0}(s)$. Since the left-hand expression $\to 0$ as $s \to \infty$, this can only happen when $c_{\nu_0\mu_0} \equiv 0$, which in turn implies that $d(s)^{mt-1}R(s) = 0$ $(s \geq s_1)$, and so

$$d(s)^{mt-1}A(s)^{1/m} = \sum_{i=1}^{N} u_i(s)\big(A_K^{1/m}\gamma_i\big)^s \quad (s \geq s_1). \tag{18}$$

That the real exponential polynomial

$$d(x)^{(mt-1)m}A(x) - \left(\sum_{i=1}^{N} u_i(x)\big(A_K^{1/m}\gamma_i\big)^x\right)^m$$

has infinitely many integer zeros, shows then that it must vanish identically which proves the theorem. □

3. Rational exponential polynomials, REP

In the last two sections, we have seen that the two main results concerning quotient and power of a real exponential polynomials hold with certain technical requirements arising from the proofs. Our aim in this

section is to try to remove these technical requirements using techniques appearing in [8]. It is convenient now to introduce the notion of *rational exponential polynomials*. An RB-EP $A(x)$ of the form (11), (12) is called a rational exponential polynomial, or REP for short, if all its polynomial coefficients $a_k(x)$ have rational coefficients. We begin with a useful deduction due to [8, Lemma 2, p. 6].

Lemma 2. *Let*

$$A(x) = a_1(x)A_1^x + \cdots + a_K(x)A_K^x$$

be an RB-EP. If $A(s) \in \mathbb{Z}$ for all $s \in \mathbb{N}$, then all exponent bases, A_k, are integral and all polynomial coefficients, $a_k(x)$, have rational coefficients, i.e., $A(x)$ is an REP with integer exponent bases.

Proof. Since all exponent bases A_k are rational, let them all be the roots of

$$\mathcal{C}(X) = c_r \prod_{k=1}^{K} (X - A_k)^{e_k} \in \mathbb{Z}[X], \quad e_k = \deg a_k + 1.$$

Let

$$\mathcal{C}_k(X) := \frac{\mathcal{C}(X)}{X - A_k}, \quad a_k(x) := b_{e_k-1,k}\, x^{e_k-1} + \cdots + b_{0,k} \quad (k = 1, \ldots, K),$$

where the $c_r \in \mathbb{Z}$ is so chosen that all $\mathcal{C}_k(X)$ and $\mathcal{C}(x)$ are in $\mathbb{Z}[X]$. From (**B**), for $s \in \mathbb{N}$ we know that the operator $\mathcal{C}_k(E)$ annihilates all the terms of $A(s)$ except the term $b_{e_k-1,k}\, s^{e_k-1}$. Now

$$\mathcal{C}_k(E)A(s) = A_k^s \, [\mathcal{C}_k(E)A(s)]_{s=0} = b_{e_k-1,k}\, A_k^s \, A_k^{e_k-1} \, \mathcal{C}_k^{(e_k-1)}(A_k). \quad (19)$$

Since the left-hand expression of (19) is an integer for all $s \in \mathbb{N}$, we have $b_{e_k-1,k} \in \mathbb{Q}$ and by considering sufficiently large s, we deduce that $A_k \in \mathbb{Z}$. The desired result follows by similar arguments. □

3.1. **Quotient.** Using Lemma 2 and a technique in [8, Corollary, p. 7], the factor $b_J(x)$ can be removed from the conclusion of Theorem 3 as we now show.

Theorem 5. *Let*

$$A(x) = a_1(x)A_1^x + \cdots + a_K(x)A_K^x, \quad 0 < A_1 < \cdots < A_K$$

with all $A_k \in \mathbb{Q}$, and

$$B(x) = b_1(x)B_1^x + \cdots + b_J(x)B_J^x, \quad 0 < B_1 < \cdots < B_J$$

with all $B_k \in \mathbb{Q}$, be two RB-EP's. For all $s \in \mathbb{N}$, if $b_J(s)$, $A(s)$ and $B(s)$ $(\neq 0)$ are integers, and $B(s) \mid A(s)$, then $A(x)/B(x)$ is an REP.

Proof. By Lemma 2, we know that both $A(x)$ and $B(x)$ are REP's. Using the notation in the proof of Theorem 3, we know that $d_i(x) \in \mathbb{Q}[x]$ $(i = 1, \ldots, t)$ and so (13) yields

$$n^s \frac{A(s)}{B(s)} = \sum_{i=1}^{t} \frac{d_i(s)}{b_J(s)^N} (n\gamma_i)^s, \tag{20}$$

where $\gamma_i \in \mathbb{Q}$, and $n \in \mathbb{N}$ is so chosen that $n\gamma_i \in \mathbb{N}$ $(i = 1, \ldots, t)$. Let $G(x) \in \mathbb{Z}[x] \setminus \{0\}$ be the l.c.m. of the denominators of $\frac{d_1(x)}{b_J(x)^N}, \ldots, \frac{d_t(x)}{b_J(x)^N}$, so that we can write

$$G(x) \frac{d_i(x)}{b_J(x)^N} =: H_i(x) \in \mathbb{Z}[x] \quad (i = 1, \ldots, t),$$

with $\gcd(H_1(x), \ldots, H_t(x), G(x)) = 1$. Suppose that $\deg G > 0$. Then there exist $T_1(x), \ldots, T_t(x), T(x) \in \mathbb{Z}[x]$ and $R \in \mathbb{Z} \setminus \{0\}$ such that

$$T_1(x)H_1(x) + \cdots + T_t(x)H_t(x) + T(x)G(x) = R.$$

We know from [9, Exercise 108, p. 131] that for sufficiently large $m_1 \in \mathbb{N}$, there is a large prime p, which can be so chosen that

$$p > \max\{R, n\gamma_1, \ldots, n\gamma_t\},$$

and $p \mid G(m_1)$. Since $A(m_1)/B(m_1) \in \mathbb{Z}$, we get for any $u \in \mathbb{N}$,

$$\begin{aligned} &\sum_{i=1}^{t} H_i(m_1 + up)(n\gamma_i)^{m_1+up} \\ &\quad = G(m_1 + up) \sum_{i=1}^{t} \frac{d_i(m_1 + up)}{b_J(m_1 + up)^N} (n\gamma_i)^{m_1+up} \\ &\quad = G(m_1 + up)\, n^{m_1+up} \frac{A(m_1 + up)}{B(m_1 + up)} \equiv 0 \pmod{p} \end{aligned}$$

and so

$$\sum_{i=1}^{t} H_i(m_1)(n\gamma_i)^{m_1}(n\gamma_i)^{up} \equiv 0 \pmod{p} \qquad (u = 0, 1, \ldots, t-1). \tag{21}$$

Considering (21) as a system of t linear congruences in the t unknowns $H_i(m_1)(n\gamma_i)^{m_1}$, whose determinant is

$$\prod_{i<j} ((n\gamma_i)^p - (n\gamma_j)^p) \equiv \prod_{i<j} (n\gamma_i - n\gamma_j) \not\equiv 0 \pmod{p},$$

we must have $H_i(m_1) \equiv 0 \pmod{p}$ $(i = 1, \ldots, t)$, which in turn implies that $p \mid R$. This contradiction shows then that $\deg G = 0$, i.e., $G(x)$ is a nonzero integer constant, and so each $d_i(x)/b_J(x)^N \in \mathbb{Q}[x]$. $\square$

3.2. **Powers.** In Theorem 4, if the coefficients $a_k(x)$ $(k = 1, \ldots, K-1)$ are polynomials with rational coefficients, i.e., $A(x)$ is an REP, we can show moreover that the base elements of the resulting exponential polynomial in the conclusion of Theorem 4 are also rational and the factor $d(x)^{(mt-1)m}$ in the conclusion can be discarded.

Theorem 6. *Let $m \in \mathbb{N}$, $m \geq 2$, and let*

$$A(x) = \sum_{k=1}^{K-1} a_k(x) A_k^x + d(x)^m A_K^x, \quad 0 < A_1 < \cdots < A_K,$$

with all $A_i \in \mathbb{Q}$, be an REP. Assume that $A(s)$ is an m^{th} power of an integer for every $s \in \mathbb{N}$. Then $A(x)$ is an m^{th} power of an REP whose base elements are integers.

Proof. Since all the polynomials $a_k(x)$ $(k = 1, \ldots, K-1)$ and $d(x)$ have rational coefficients, let $q \in \mathbb{N}$ be such that

$$q^m A(x) = \sum_{k=1}^{K-1} \left(q^m a_k(x)\right) A_k^x + (qd(x))^m A_K^x$$

is an REP all of whose coefficients are polynomials with integer coefficients. Following the proof of Theorem 4, with $q^m A(x)$, $qd(x)$ in place of $A(x)$ and $d(x)$, respectively, up to (18), we get

$$(qd(s))^{mt-1}\, qA(s)^{1/m} = \sum_{i=1}^{N} u_i(s) \left(A_K^{1/m} \gamma_i\right)^s \quad (s \geq s_1),$$

where all the coefficient polynomials $u_i(x)$ now have rational coefficients. Taking s sufficiently large of the form $ms' + 1$ $(s' \in \mathbb{N})$, we get

$$\left(qd(ms'+1)\right)^{mt-1} qA(ms'+1)^{1/m} = A_K^{1/m} \sum_{i=1}^{N} u_i(ms'+1) A_K^{s'} \gamma_i^{ms'+1},$$

which shows that $A_K^{1/m} \in \mathbb{Q}$. Since the REP

$$(qd(x))^{(mt-1)m}\ q^m A(x) - \left(\sum_{i=1}^{N} u_i(x) (A_K^{1/m} \gamma_i)^x\right)^m$$

has infinitely many integer zeros, it must vanish identically. By Lemma 2, the REP

$$(qd(x))^{mt-1}\, qA(x)^{1/m} = \sum_{i=1}^{N} u_i(x) \left(A_K^{1/m} \gamma_i\right)^x \tag{22}$$

must have all base elements $A_K^{1/m} \gamma_i$ being integral.

We proceed next to eliminate the cumbersome factor on the left-hand side of (22), we use the same technique as in the proof of Theorem 5. Rewrite (22) as

$$n^x A(x)^{1/m} = \sum_{i=1}^{N} \frac{u_i(x)}{(qd(x))^{mt-1}\, q} \left(nA_K^{1/m}\gamma_i\right)^x, \tag{23}$$

where $n \in \mathbb{N}$ is so chosen that $nA_K^{1/m}\gamma_i \in \mathbb{N}$ $(i = 1, \ldots, t)$. Let

$$g(x) \in \mathbb{Z}[x] \setminus \{0\}$$

be the l.c.m. of the denominators of

$$\frac{u_1(x)}{(qd(x))^{mt-1}q}, \quad \ldots, \quad \frac{u_t(x)}{(qd(x))^{mt-1}q}.$$

Thus, we can write

$$g(x)\frac{u_i(x)}{(qd(x))^{mt-1}q} =: h_i(x) \in \mathbb{Z}[x] \quad (i = 1, \ldots, t),$$

with $\gcd(h_1(x), \ldots, h_t(x), g(x)) = 1$. Suppose that $\deg g > 0$. Then there exist $T_1(x)$, $\ldots$, $T_t(x)$, $T(x) \in \mathbb{Z}[x]$ and $R \in \mathbb{Z} \setminus \{0\}$ such that

$$T_1(x)h_1(x) + \cdots + T_t(x)h_t(x) + T(x)g(x) = R.$$

We know from [9, Exercise 108, p. 131] that for sufficiently large $m_1 \in \mathbb{N}$, there is a large prime p, which can be so chosen that

$$p > \max\left\{R, nA_K^{1/m}\gamma_1, \ldots, nA_K^{1/m}\gamma_t\right\},$$

and $p \mid g(m_1)$. Since $A(m_1)^{1/m} \in \mathbb{Z}$, we get for any $u \in \mathbb{N}$,

$$\begin{aligned}
&\sum_{i=1}^{t} h_i(m_1 + up)\left(nA_K^{1/m}\gamma_i\right)^{m_1+up} \\
&\qquad = g(m_1 + up) \sum_{i=1}^{t} \frac{u_i(m_1 + up)}{(qd(m_1 + up))^{mt-1}q} \left(nA_K^{1/m}\gamma_i\right)^{m_1+up} \\
&\qquad = g(m_1 + up)\; n^{m_1+up} A(m_1 + up)^{1/m} \equiv 0 \pmod{p}
\end{aligned}$$

and so

$$\sum_{i=1}^{t} h_i(m_1)\left(nA_K^{1/m}\gamma_i\right)^{m_1}\left(nA_K^{1/m}\gamma_i\right)^{up} \equiv 0 \pmod{p}$$

$$(u = 0, 1, \ldots, t-1). \tag{24}$$

Considering (24) as a system of t linear congruences in the t unknowns $h_i(m_1)\left(nA_K^{1/m}\gamma_i\right)^{m_1}$, whose determinant is

$$\prod_{i<j}\left(\left(nA_K^{1/m}\gamma_i\right)^p-\left(nA_K^{1/m}\gamma_j\right)^p\right)$$
$$\equiv\prod_{i<j}\left(nA_K^{1/m}\gamma_i-nA_K^{1/m}\gamma_j\right)\not\equiv 0 \pmod p,$$

we must have $h_i(m_1)\equiv 0 \pmod p$ $(i=1,\dots,t)$, which in turn implies that $p \mid R$. This contradiction shows then that $\deg g = 0$, i.e., $g(x)$ is a nonzero integer constant, and so each $u_i(x)/(qd(x))^{mt-1}q \in \mathbb{Q}[x]$. Thus, (23) shows that $A(x)^{1/m}=\sum_{i=1}^N \frac{u_i(x)}{(qd(x))^{mt-1}q}\left(A_K^{1/m}\gamma_i\right)^x$ is an REP. The desired conclusion now follows immediately from Lemma 2. □

There remains one last technical assumption in Theorems 2, 4 and 6 that the leading coefficient $a_K(x)$ is a perfect m^{th} power. This restriction can be removed using another reduction technique from [8, pp. 14–15]. This technique needs quite a few preparations. Denote by $\mathcal{A}$ the following integral domain

$$\mathcal{A}:=\left\{f:\mathbb{N}_0\to\mathbb{Z}\ \mid\ f(x)=\sum_{j=1}^M P_j(x)\rho_j^x,\ P_j\in\mathbb{Z}[x],\ \rho_j\in\mathbb{N},\ M\in\mathbb{N}\right\}.$$

For $f\in\mathcal{A}$ as above, define $\deg f := \max_j \deg P_j$. Let $\mathcal{E}$ be a subring of $\mathcal{A}$ defined by

$$\mathcal{E}:=\{f\in\mathcal{A}\mid \deg f=0\}.$$

The next lemma is modified from [8, pp. 15–16].

Lemma 3. *Let $m\in\mathbb{N}$, $m\ge 2$ and let*

$$f(x)=\sum_{j=1}^M P_j(x)\rho_j^x\in\mathcal{A},\quad \rho_1>\rho_2>\cdots>\rho_M\ge 1.$$

If $f(s)\in\mathbb{Z}^m$ for all $s\in\mathbb{N}$, then there are $d\in\mathbb{N}$, $B(x)\in\mathbb{Z}[x]$ such that $d\cdot P_1(x)=B(x)^m$.

Proof. Let

$$F_s(x)=\sum_{j=1}^M P_j(s-x)\rho_j^s.$$

Considering both $F_s(x)$ and $F_s'(x)$ as polynomials in x and using Euclid's algorithm, which involves only rational operations, we get

$$R_s(x):=\gcd\left(F_s(x),F_s'(x)\right)=x^n+C_{n-1,s}\,x^{n-1}+\cdots+C_{0,s}, \tag{25}$$

where $C_{j,s}\in\mathcal{K}_A$, the quotient field of A, $(j=0,1,\dots,n-1)$. Let

$$F_s(x)=F(s)\prod_{j=1}^L (x-\beta_{j,s})^{v_{j,s}},\quad \beta_{i,s}\ne\beta_{j,s}\quad (i\ne j).$$

By [8, Lemma 8, p. 14], we know that $v_{j,s} \geq m \geq 2$, and so

$$R_s(x) = \prod_{j=1}^{L} (x - \beta_{j,s})^{v_{j,s}-1}$$

yielding

$$W_s(x) := \frac{F_s(x)}{R_s(x)} = F(s) \prod_{j=1}^{L} (x - \beta_{j,s}) \,. \tag{26}$$

We see that $W_s(x)$ is of the same form as in (25). By [8, Lemma 8], we know that $W_s(x)^m \mid F_s(x)$ and their quotient $Q_s(x) := F_s(x)/W_s(x)^m$ is again of the form (25). Multiplying $F_s(x) = W_s(x)^m Q_s(x)$ by a common denominator of the coefficients of the form $D(s)^m$, with $D(s) \in \mathcal{A}$, we get

$$D(s)^m F_s(x) = B_s(x)^m T_s(x), \tag{27}$$

where $B_s(x), T_s(x) \in \mathcal{A}[x]$.

Let $\bar{D}(s) := \gcd\left(D(s), B_s(x)\right)$ computed in $\mathcal{K}_A[x]$. Then we have

$$\frac{D(s)^m}{\bar{D}(s)^m} F_s(x) = \frac{B_s(x)^m}{\bar{D}(s)^m} T_s(x),$$

$$\tilde{D}(s)^m F_s(x) = \mathcal{B}_s(x)^m T_s(x).$$

So,

$$F_s(x) = \mathcal{B}_s(x)^m \frac{T_s(x)}{\tilde{D}(s)^m} = \mathcal{B}_s(x)^m \mathcal{T}_s(x), \tag{28}$$

where $\tilde{D}(s) = D(s)/\bar{D}(s) \in \mathcal{A}$, $\mathcal{B}_s(x) = B_s(x)/\bar{D}(s) \in \mathcal{A}[x]$ and $\mathcal{T}_s(x) = T_s(x)/\bar{D}(s)^m \in \mathcal{A}[x]$. Since, from (26), $\mathcal{B}_s(x)$ contains $W_s(x)$ as a factor, we see that $\deg_x \mathcal{B}_s(x) \geq 1$, and so $\deg_x \mathcal{T}_s(x) < \deg_x F_s(x)$. As elaborated in [12], since $F_s(x)$ is a polynomial in $s - x$ with coefficients in $\mathcal{E}$, any factorization preserves this property, i.e., $\mathcal{T}_s(x)$ is also a polynomial in $s - x$ with coefficients in $\mathcal{E}$, and so its degree in $s - x$, which equals its degree in x must be $< \deg_x F_s(x)$.

Putting $x = 0$ in (28), and noting that $F_s(0) = f(s)$, we have

$$f(s) = \mathcal{B}_s(0)^m \, \mathcal{T}_s(0) := \mathcal{B}(s)^m \, \mathcal{T}(s),$$

where $\mathcal{B}_s(0) = \mathcal{B}(s) \in \mathcal{A}$, and $\mathcal{T}_s(0) = \mathcal{T}(s) \in \mathcal{A}$. For each $\tilde{s} \in \mathbb{N}$, since $\mathcal{B}(\tilde{s}) \in \mathbb{Z}$ and $f(\tilde{s}) \in \mathbb{Z}^m$, we have $\mathcal{T}(\tilde{s}) \in \mathbb{Z}^m$.

Repeating the step with $\mathcal{T}_s(x)$ in place of $F_s(s)$, we get a similar relation but with the second factor on the right-hand side being of lower degree. After a finite number of such repeated reductions, we end up with an identity of the form

$$F_s(x) = \tilde{\mathcal{B}}_s(x)^m \, \tilde{\mathcal{T}}_s(x),$$

where $\tilde{\mathcal{T}}_s(x)$ is of degree 0, i.e., $\tilde{\mathcal{T}}_s(x) \in \mathcal{E}$ and $\tilde{\mathcal{B}}_s(x) \in \mathcal{A}[x]$. Thus,

$$f(s) = F_s(0) = \tilde{\mathcal{B}}_s(0)^m \, \tilde{\mathcal{T}}_s(0) = \tilde{\mathcal{B}}(s)^m \tilde{\mathcal{T}}(s),$$

where the two factors on the right-hand side are of the form

$$\tilde{\mathcal{B}}(s) = \sum_{j=1}^{M_B} B_j(s)\, b_j^s \in \mathcal{A} \quad (b_1 > b_2 > \cdots > b_{M_B}),$$

$$\tilde{\mathcal{T}}(s) = \sum_{j=1}^{M_T} T_j\, t_j^s \in \mathcal{E} \quad (t_1 > t_2 > \cdots > t_{M_T}).$$

Equating the highest term, we obtain $P_1(s) = T_1\, B_1(s)^m$. Multiplying through by T_1^{m-1}, we get

$$T_1^{m-1} P_1(s) = (T_1\, B_1(s))^m = B(s)^m,$$

and the lemma is proved. □

Using this last lemma, we can now finish our task.

Theorem 7. *Let $m \in \mathbb{N}$, $m \geq 2$, and let*

$$A(x) = \sum_{k=1}^{K} a_k(x) A_k^x, \quad 0 < A_1 < \cdots < A_K$$

with all $A_i \in \mathbb{Q}$, be an REP. If $A(s)$ is an m^{th} power of an integer for every $s \in \mathbb{N}$, then there exists $d \in \mathbb{N}$ such that $d \cdot A(x)$ is an m^{th} power of another REP.

Proof. Let $q, n \in \mathbb{N}$ be such that

$$(qn^x)^m \, A(x) = \sum_{k=1}^{K} q^m a_k(x) (n^m A_k)^x \in \mathcal{A}$$

is an REP whose base elements $n^m A_k$ are integers and whose coefficients $q^m a_k(x)$ are polynomials with integer coefficients. By Lemma 3, there are $D \in \mathbb{N}$, $B(x) \in \mathbb{Z}[x]$ such that $Dq^m a_K(x) = B(x)^m$, i.e.,

$$D\,(qn^x)^m \, A(x) = \sum_{k=1}^{K-1} Dq^m a_k(x) (n^m A_k)^x + B(x)^m (n^m A_K)^x.$$

By Theorem 6, we deduce that $D\,(qn^x)^m \, A(x)$ is an m^{th} power of an REP whose base elements are integers, and the result follows. □

4. Acknowledgements

The first author wishes to thank the organizing and the program committees of the Sixth International Conference on Analytic Number Theory and Spatial Tessellations for their invitation and hospitality, and to thank the Department of Mathematics, Faculty of Science, Kasetsart University for financial support.

References

1. J. W. S. Cassels, *An Introduction to Diophantine Approximation*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 45, Cambridge, 1957.
2. H. Davenport, D. J. Lewis and A. Schinzel, *Polynomials of certain special types*, Acta Arith. **9** (1964), 107–116.
3. T. Kojima, *Note on number-theoretic properties of algebraic functions*, Tôhoku Math. J. **8** (1915), 24–37.
4. W. J. LeVeque, *On the equation* $y^m = f(x)$, Acta Arith. **IX** (1964), 209–219.
5. D. J. Lewis and P. Morton, *Quotients of polynomials and a theorem of Pisot and Cantor*, J. Fac. Sci. Univ. Tokyo **28** (1982), 813–822.
6. L. Lovász, *On finite Dirichlet series*, Acta Math. Acad. Sci. Hungarica **22** (1971), 227–231.
7. A. Perelli and U. Zannier, *A note on arithmetic properties of certain recurring sequences*, Colloquia Mathematica Societatis János Bolyai, **34**. Topics in Classical Number Theory, Budapest (Hungary), 1981, pp. 1209–1215.
8. A. Perelli and U. Zannier, *Arithmetic properties of certain recurrence sequences*, J. Austral. Math. Soc. **37** (Ser. A) (1984), 4–16.
9. G. Pólya and G. Szegö, *Problems and Theorems in Analysis*, Vol. II, Springer, 1976.
10. P. Ribenboim, *Polynomials whose values are powers*, J. Reine Angew. Math. **268/269** (1974), 34–40.
11. J. F. Ritt, *On the zeros of exponential polynomials*, Trans. Amer. Math. Soc. **31** (1929), 680–686.
12. R. S. Rumely and A. J. van der Poorten, A note on the Hadamard kth root of a rational function, J. Austral. Math. Soc. (Ser. A) **43** (1987), 314–327.
13. T. N. Shorey and R. Tijdeman, *Exponential Diophantine equations*, Cambridge Tracts in Mathematics No. **87**, Cambridge, 1986.
14. M. Voorhoeve and A. J. van der Poorten, *Wronskian determinants and the zeros of certain functions*, Proc. Kon. Akad. v. Wetensch., Ser. A. **78** (1975), 417–424.

Department of Mathematics, Faculty of Science, Kasetsart University, Bangkok, 10900, Thailand
Email: fscivil@ku.ac.th,fscipar@ku.ac.th

Voronoï's Impact on Modern Science, Book 6, Volume 1
Proceedings of the Sixth International Conference on Analytic Number Theory and Spatial Tessellations

STEINITZ'S NUMBERS AND DIAGONAL CONSTRUCTIONS OF DISCRETE STRUCTURES

Bogdana Oliynyk (*Ukraine*)

Abstract. In this paper some properties of Steinitz or supernatural numbers, that are a generalization of positive integers, are considered. It is proved that there are countable many sublattices of Steinitz's lattice that are isomorphic to the Boolean algebra of subsets of a countable set. The diagonal limits of Hamming spaces over some finite alphabet and diagonal limits of their isometry groups are characterized in terms of the lattice of Steinitz numbers.

Introduction

The concept of a Steinitz or supernatural number was introduced by E. Steinitz in the Crelle's Journal in 1910 [1].

Let $\mathbb{P}$ be the set of all primes. A *supernatural number* or *Steinitz number* is an infinite formal product of the form

$$\prod_{p\in\mathbb{P}} p^{k_p},$$

where $k_p \in \mathbb{N}\cup\{0,\infty\}$. We denote by a symbol $\mathbb{SN}$ the set of all supernatural numbers. Obviously, any positive integer is a Steinitz number. The elements of the set $\mathbb{SN}\setminus\mathbb{N}$ are called *infinite supernatural numbers*.

The product of supernatural numbers

$$\prod_{p\in\mathbb{P}} p^{r_p} \quad \text{and} \quad \prod_{p\in\mathbb{P}} p^{k_p}$$

is the supernatural number

$$\prod_{p\in\mathbb{P}} p^{r_p+k_p},$$

Received November 20, 2018.

2010 *Mathematics Subject Classification.* Primary 03G10; Secondary 46A13, 54E40.

Key words and phrases. Steinitz number, Steinitz's lattice, diagonal embedding, Hamming space, monomial group.

where $t + \infty = \infty + t = \infty + \infty = \infty$ for all $t \geq 0$.

A supernatural number v divides a supernatural number u if there exists $w \in \mathbb{SN}$ such that $u = v \cdot w$. The divisibility relation $|$ transforms $\mathbb{SN}$ into a partially ordered set with the greatest element

$$\Omega = \prod_{p \in \mathbb{P}} p^{\infty}$$

and the least element 1.

The poset $(\mathbb{SN}, |)$ is a lattice and it is called *the Steinitz's lattice*. The supremum and the infimum of supernatural numbers u and v are defined by the equalities

$$u \vee v = \prod_{p \in \mathbb{P}} p^{\max(k_p, l_p)},$$

$$u \wedge v = \prod_{p \in \mathbb{P}} p^{\min(k_p, l_p)},$$

where $\max(k, \infty) = \infty$, $\min(k, \infty) = k$ for $k \in \mathbb{N} \cup \{0\}$.

The Steinitz's lattice $(\mathbb{SN}, \wedge, \vee)$ is a complete distributive lattice. It suits to characterize various mathematical objects. In particular, the Steinitz's lattice is isomorphic to:

- the lattice of all subfields of an algebraic closure of a finite field [2];
- the lattice of Glimm's subalgebras of a limited C^*-algebra [3, 4];
- lattices of homogeneous symmetric and homogeneous alternating subgroups of the symmetric groups of natural numbers [5];
- lattices of Hamming spaces of periodic sequences over the alphabet $\{0, 1\}$ [6];
- the lattice of periodically defined linear transformations of an infinite-dimensional vector space V over the field K [7].

In this paper we consider some properties of Steinitz's lattice. In particular, we construct a countable family of sublattices of the Steinitz's lattice such that each of them is isomorphic to the Boolean algebra of subsets of a countable set. Also we describe two types of diagonal limits of metric spaces and groups, that are naturally parametrized by Steinitz's numbers, namely, diagonal limits of Hamming spaces over some finite alphabet and diagonal limits of monomial groups. We discuss connection between them. We prove that the diagonal limit of some specific monomial groups is a subgroup of the full isometry group of a diagonal limit of Hamming spaces over a finite alphabet.

1. Subsets of Steinitz's lattice

Fix some positive integer l. We say that a supernatural number

$$u = \prod_{p \in \mathbb{P}} p^{k_p}$$

is *double-barrelled* if the degree k_p belongs to the set $\{l, \infty\}$ for all $p \in \mathbb{P}$. The set of all double-barrelled numbers is denoted by DB_l. Note that the set DB_l of double-barrelled supernatural numbers is closed with respect to the operations $\vee$, $\wedge$ and contains the numbers

$$\Omega = \prod_{p \in \mathbb{P}} p^{\infty} \quad \text{and} \quad \prod_{p \in \mathbb{P}} p^{l}.$$

Define on the set DB_l the operation of negation in the following way:

$$\bar{u} = \prod_{p \in \mathbb{P}} p^{\bar{k}_p},$$

where

$$\bar{k}_p = \begin{cases} l, & \text{if } k_p = \infty, \\ \infty, & \text{if } k_p = l. \end{cases}$$

According to the definition, a double-barrelled supernatural number u is uniquely determined by the subset $\mathcal{O}(u)$ of all primes p from $\mathbb{P}$ such that $k_p = \infty$. Then the negation $\bar{u}$ of the number $u \in DB_l$ is a double-barrelled supernatural number that is determined by subset $\mathcal{O}(\bar{u}) = \mathbb{P} \setminus \mathcal{O}(u)$. It is easy to see that the following equalities

$$u \vee \bar{u} = \Omega, \qquad u \wedge \bar{u} = \prod_{p \in \mathbb{P}} p^{l} \tag{1}$$

hold.

Theorem 1. *For any positive integer l, the set of all double-barrelled numbers DB_l with the operations $\vee$, $\wedge$, $^-$ forms a Boolean algebra with*

$$a = \prod_{p \in \mathbb{P}} p^{l},$$

as the zero element and Ω as the unit element. This Boolean algebra is isomorphic to the Boolean algebra of subsets of a countable set.

Proof. Using (1) one can directly verify that $(DB_l, \vee, \wedge, ^-)$ is a Boolean algebra with $a = \prod_{p \in \mathbb{P}} p^l$ as the zero element and Ω as the unit element. We need to show that this algebra is isomorphic to the Boolean algebra of subsets of a countable set.

Define the mapping φ from the algebra of subsets of the set $\mathbb{P}$ to the algebra DB_l as follows. For any subset $\mathcal{X} \subset \mathbb{P}$, we put $\varphi(\mathcal{X}) = u$, where $\mathcal{O}(u) = \mathcal{X}$. As any double-barrelled supernatural number u is uniquely determined by the subset $\mathcal{O}(u)$, φ is a bijection. In addition, for any $\mathcal{X}_1, \mathcal{X}_2 \subset \mathbb{P}$ such that $\varphi(\mathcal{X}_1) = u_1$, $\varphi(\mathcal{X}_2) = u_2$, we have

$$\varphi(\mathcal{X}_1 \cup \mathcal{X}_2) = u_1 \vee u_2, \qquad \varphi(\mathcal{X}_1 \cap \mathcal{X}_2) = u_1 \wedge u_2, \qquad \varphi(\bar{\mathcal{X}}_1) = \bar{u}_1.$$

Hence, the mapping φ is an isomorphism. In particular, it means that $(DB_l, \vee, \wedge, ^-)$ is a Boolean algebra. □

Since Theorem 1 is valid for any positive integer l, we immediately obtain the following corollary.

Corollary 1. *There exist countable many sublattices of the Steinitz's lattice isomorphic to the Boolean algebra of subsets of a countable set.*

Note that Theorem 1 is a generalization of Lemma 3 from [8].

2. Diagonal limits of Hamming spaces

The Hamming metric was introduced in information theory as the most common tool to measure the difference between binary strings of the same length.

Let $B = \{b_1, \ldots, b_q\}$ be an alphabet, $q \geq 2$. Denote by $H_n(q)$ the *Hamming space* of dimension n on alphabet B. This space consists of all n-tuples $(a_1, \ldots, a_n)$, $a_i \in B$, $1 \leq i \leq n$, where the distance d_{H_n} between two n-tuples is equal to the number of coordinates where they differ.

The *scaled Hamming space* $\hat{H}_n(q)$ is defined on the same set of all n-tuples $(a_1, \ldots, a_n)$, $a_i \in B$, $1 \leq i \leq n$, but the distance is determined as $\frac{1}{n} d_{H_n}$. Without loss of generality we assume that scaled Hamming spaces are defined over the alphabet $B = \{0, \ldots, q-1\}$.

Let now $\tau = (m_1, m_2, \ldots)$ be a *divisible* sequence of positive integers, that is $m_i \mid m_{i+1}$ for all $i \in \mathbb{N}$. We denote by $(s_1, s_2, \ldots)$ the sequence of ratios of the sequence τ, i.e.

$$s_1 = m_1, \quad s_{i+1} = \frac{m_{i+1}}{m_i}, \quad i \geq 1.$$

The supernatural number

$$s_1 \cdot s_2 \cdot s_3 \cdot \ldots$$

is called the *characteristic of sequence* τ and denoted by $\mathrm{char}(\tau)$. It is easy to see that, if the sequence τ is strictly increasing divisible sequence, then $\mathrm{char}(\tau)$ is an infinite supernatural number.

For any $i \geq 1$, define an isometric embedding

$$\psi_{s_i} : \hat{H}_{m_i}(q) \to \hat{H}_{m_{i+1}}(q)$$

by the rule:

$$\psi_{s_i}(x_1, \ldots, x_{m_i}) = (\underbrace{x_1, \ldots, x_{m_i} \mid x_1, \ldots, x_{m_i} \mid \ldots \mid x_1, \ldots, x_{m_i}}_{s_i \cdot m_i}).$$

Then the sequence τ determines the directed system of scaled Hamming spaces on the alphabet B

$$\langle \hat{H}_{m_i}(q), \psi_{s_i} \rangle_{i \in \mathbb{N}}$$

with the diagonal embeddings ψ_{s_i}, $i \geq 1$.

The limit space of the directed system

$$\mathcal{H}(\tau, q) = \varinjlim \langle \hat{H}_{m_i}(q), \psi_{s_i} \rangle$$

is called a *diagonal limit* of spaces $\hat{H}_{m_i}$.

Proposition 1 ([9])**.** *Let τ_1, τ_2 be increasing divisible sequences. Then the spaces $\mathcal{H}(\tau_1, q)$ and $\mathcal{H}(\tau_2, q)$ are isometric iff* $\mathrm{char}(\tau_1) = \mathrm{char}(\tau_2)$.

We can describe spaces $\mathcal{H}(\tau, q)$ as some subspaces of the Besicovitch space. Let us recall some definitions. On the set $B^{\mathbb{N}} = \{0, \ldots, q-1\}^{\mathbb{N}}$ of all infinite sequences over B determine a pseudo-metric $\hat{d}_B$ as follows. For arbitrary sequences $\mathbf{x} = (x_1, x_2, \ldots)$ and $\mathbf{y} = (y_1, y_2, \ldots)$ from $B^{\mathbb{N}}$, let

$$\hat{d}_B(x, y) = \limsup_{n \to \infty} \frac{1}{n} d_{H_n}\big((x_1, \ldots, x_n), (y_1, \ldots, y_n)\big).$$

The pseudo-metric $\hat{d}_B$ defines an equivalence $\sim_{\hat{d}_B}$ on $B^{\mathbb{N}}$, i.e.,

$$\mathbf{x} \sim_{\hat{d}_B} \mathbf{y} \quad \text{if and only if} \quad \hat{d}_B(\mathbf{x}, \mathbf{y}) = 0.$$

Denote by $\mathcal{X}_B = B^{\mathbb{N}} / \sim_{\hat{d}_B}$ the quotient set by this equivalence. The function $\hat{d}_B$ induces the metric d_B on $\mathcal{X}_B$. The metric space $(\mathcal{X}_B, d_B)$ is called the *Besicovitch space* [10]. It is well known that the metric space $(\mathcal{X}_B, d_B)$ is complete, nonseparable, and is not locally compact.

We now show that each space $\mathcal{H}(\tau, q)$ is isometric to some subspace of the Besicovitch space.

An infinite sequence $\mathbf{a} = (a_1, a_2, \ldots)$ is said to be *periodic* if there exists a natural number k such that the equality

$$a_i = a_{i+k}$$

holds for all $i \in \mathbb{N}$. In this case the number k is called a *period* of the sequence $\mathbf{a}$. A periodic sequence $\mathbf{a}$ is called *u-periodic* for some supernatural number u if its minimal period divides u.

Let u be some infinite supernatural number. Denote by $\mathcal{H}(u)$ the subspace of the Besicovitch space $(\mathcal{X}_B, d_B)$ consisting of all u-periodic sequences over the set B. We call the metric space $\mathcal{H}(u)$ the *u-periodic Hamming space*.

Proposition 2. *Let $\tau = (m_1, m_2, \ldots)$ be a divisible sequence of positive integers. The diagonal limit $\mathcal{H}(\tau, q)$ of spaces $\hat{H}_{m_i}$, $i \geq 1$, is isometric to the u-periodic Hamming space $\mathcal{H}(u)$, where*

$$u = s_1 \cdot s_2 \cdot s_3 \ldots,$$

and the sequence $(s_1, s_2, s_3, \ldots)$ is the sequence of ratios of the sequence τ.

Proof. Define a function ϕ from $\mathcal{H}(\tau, q)$ to the u-periodic Hamming space $\mathcal{H}(u)$. From the definition of the diagonal limit of Hamming spaces it follows that for any point x from $\mathcal{H}(\tau, q)$ there exists an index i, $i \geq 1$, such that x belongs to $\hat{H}_{m_i}(q)$. Then $x = (x_1, \ldots, x_{m_i})$ and we define

$$\phi(x) = (x_1, \ldots, x_{m_i} \mid x_1, \ldots, x_{m_i} \mid \ldots \mid x_1, \ldots, x_{m_i}, \ldots).$$

It is clear that the function ϕ is a bijection and preserves distances. Hence, $\mathcal{H}(\tau, q)$ and $\mathcal{H}(u)$ are isometric. □

Note that from Proposition 1 and Proposition 2 immediately follows that, for different supernatural numbers u_1 and u_2, subspaces $\mathcal{H}(u_1)$ and $\mathcal{H}(u_2)$ are not isometric.

Theorem 2. *A family of subspaces $\mathcal{H}(u)$, $u \in \mathbb{SN}$, of the space $\mathcal{X}_B$ forms a lattice with respect to inclusion which is isomorphic to the lattice of supernatural numbers.*

Proof. Since each periodic Hamming space can be described as some relational structure, the statement of this theorem follows from Theorem 1 of [8]. □

3. Isometries of diagonal limits of Hamming spaces

The isometry group Isom $\hat{H}_n(q)$ of the metric space $\hat{H}_n(q)$ is well studied. It is isomorphic to the wreath product $W(n, q) = S_q \wr S_n$, $n \geq 1$. Such a group is a particular case of monomial groups $W(n, q) = H \wr S_n$, where H is a group and S_n is the symmetric group defined on the set $\{1, \ldots, n\}$. For details on monomial groups, see, for example, [11, 12].

Let $\{x_1, \dots, x_n\}$ be some set of values. Assume that $g_1, \dots, g_k$ and $h_1, \dots, h_k$ are some permutations of the group S_q. The *monomial permutation* on the group S_q is a transformation of the form

$$\begin{pmatrix} x_1 & x_2 & \dots & x_n \\ x_{i_1}g_1 & x_{i_2}g_2 & \dots & x_{i_n}g_n \end{pmatrix} = \left(\begin{array}{c} x_k \\ x_{i_k}g_k \end{array} \middle| \, 1 \le k \le n \right). \tag{2}$$

Two monomial permutations

$$\pi = \left(\begin{array}{c} x_k \\ x_{i_k}g_k \end{array} \middle| \, 1 \le k \le n \right) \quad \text{and} \quad \gamma = \left(\begin{array}{c} x_k \\ x_{j_k}h_k \end{array} \middle| \, 1 \le k \le n \right)$$

are multiplied by the rule:

$$\pi\gamma = \left(\begin{array}{c} x_k \\ x_{i_{j_k}}g_k h_{j_k} \end{array} \middle| \, 1 \le k \le n \right). \tag{3}$$

The inverse transformation to the permutation π is

$$\pi^{-1} = \left(\begin{array}{c} x_{i_k} \\ x_k g_k^{-1} \end{array} \middle| \, 1 \le k \le n \right). \tag{4}$$

The *complete monomial group of degree n of* S_q is a group of all monomial permutations (2) with operation (3). We denote this group by $W(n, q)$.

Let $\tau = (m_1, m_2, \dots)$ be divisible sequence of positive integers. For some

$$\gamma = \begin{pmatrix} x_1 & x_2 & \dots & x_{m_i} \\ h_1 x_{j_1} & h_2 x_{j_2} & \dots & h_{m_i} x_{j_{m_i}} \end{pmatrix},$$

we define an embedding

$$\begin{array}{rccc} \phi^{m_{i+1}} : & W(m_i, q) & \to & W(m_{i+1}, q) \\ & \gamma & \to & \phi^{m_{i+1}}(\gamma) \end{array}$$

$\phi^{m_{i+1}}(\gamma) =$

$$\left(\begin{array}{ccc|ccc|c} x_1 & \dots & x_{m_i} & x_{m_i+1} & \dots & x_{2m_i} & \dots \\ h_1 x_{j_1} & \dots & h_{m_i} x_{j_{m_i}} & h_1 x_{m_i+j_1} & \dots & h_{m_i} x_{m_i+j_{m_i}} & \dots \end{array} \right. \left| \begin{array}{c} x_{(s_{i+1}-1)m_i+m_i} \\ h_{m_i} x_{(s_{i+1}-1)m_i+m_{j_{m_i}}} \end{array} \right),$$

where $m_{i+1} = m_i s_{i+1}$.

Then we obtain a directed system defined by the following embeddings:

$$\{1\} \overset{\phi^{s_1}}{\to} W(m_1, q) \overset{\phi^{s_2}}{\to} W(m_2, q) \overset{\phi^{s_3}}{\to} W(m_3, q) \overset{\phi^{s_4}}{\to} \dots.$$

The diagonal limit $W(\tau, q)$ is called the complete monomial group.

Hence, for a divisible sequence of positive integers $\tau = (m_1, m_2, \dots)$, we obtain the directed system of scaled Hamming spaces

$$\{1\} \overset{\psi^{s_1}}{\to} \hat{H}_{m_1}(q) \overset{\psi^{s_2}}{\to} \hat{H}_{m_2}(q) \overset{\psi^{s_3}}{\to} \hat{H}_{m_3}(q) \overset{\psi^{s_4}}{\to} \dots$$

and the directed system of its isometry groups

$$\{1\} \overset{\phi^{s1}}{\to} W(m_1, q) \overset{\phi^{s2}}{\to} W(m_2, q) \overset{\phi^{s3}}{\to} W(m_3, q) \overset{\phi^{s4}}{\to} \dots.$$

Theorem 3. *Let $\tau = (m_1, m_2, \dots)$ be a divisible sequence of positive integers. Then the complete monomial group $W(\tau, q)$ is a subgroup of the isometry group of the diagonal limit $\mathcal{H}(\tau, q)$ of Hamming spaces.*

Proof. Let π be an element of the group $W(\tau, q)$, x be some point of $\mathcal{H}(\tau, q)$. Since $\pi \in W(\tau, q)$, there exists a positive integer r such that we may assume that $\pi \in W(r, q)$. Similarly, there exists a positive n such that $x \in \hat{H}_n(q)$. Note that $W(\tau, q)$ and $\mathcal{H}(\tau, q)$ are defined by the same divisible sequence τ. Hence, there exists a positive integer m such that one of equalities holds: $mr = n$ or $mn = r$.

In the first case we can claim that $\pi \in W(r, q)$ and π acts on all elements $\hat{H}_n(q)$ as isometry.

In the second case we can assume that $x \in \hat{H}_r(q)$ and π acts on all elements $\hat{H}_r(q)$ as isometry.

Therefore, π is a bijection. Moreover, π preserves the number of different coordinates between two different points $x, y \in \mathcal{H}(\tau, q)$ in some common period. Hence, from the definition of the distance in the Besicovitch space it follows that π does not change distances between points of the space $\mathcal{H}(\tau, q)$. Therefore, π is an isometry of $\mathcal{H}(\tau, q)$. □

Larger classes of monomial groups are studied in [13]. Any complete monomial group $W(\tau, q)$ may be indexed by some Steinitz number $\operatorname{char} \tau$. We denote the complete monomial group $W(\tau, q)$ by $W(u, q)$, where

$$u = \operatorname{char} \tau.$$

Theorem 2.11 of [13] directly implies the following property of groups $W(u, q)$, $u \in \mathbb{SN}$.

Theorem 4. *The family $\{W(\tau, q) : q \in \mathbb{SN}\}$ forms a complete lattice with respect to set inclusion. This lattice is isomorphic to the lattice of Steinitz numbers $(\mathbb{SN}, |)$.*

References

1. E. Steinitz, *Algebraische Theorie der Körper* [*Algebraic theory of fields*], J. Reine Angew. Math. **137** (1910), 167–309; reprinted in *Algebraische Theorie der Körper*, Chelsea Publ., New York, 1950.
2. J. V. Brawley and G. E. Schnibben, *Infinite algebraic extensions of finite fields*, Contemp. Math., vol. 95, Amer. Math. Soc., Providence, RI, 1989.
3. J. G. Glimm, *On a certain class of operator algebras*, Trans. Amer. Math. Soc. **95** (1960), no. 2, 318–340.

4. S. C. Power, *Limit algebras: an introduction to subalgebras of C^*-algebras*, Pitman Res. Notes Math. Ser., vol. 278, Longman Sci. Tech., Harlow, and Wiley, New York, 1993.
5. N. V. Kroshko and V. I. Sushchansky, *Direct limits of symmetric and alternating groups with strictly diagonal embeddings*, Arch. Math. (Basel) **71** (1998), no. 3, 173–182.
6. B. V. Oliynyk and V. I. Sushchanskiĭ, *Группы изометрий пространств Хемминга периодических последовательностей* [*The isometry groups of the Hamming spaces of periodic sequences*], Сиб. мат. журн. **54** (2013), no. 1, 163–179 = Sib. Math. J. **54** (2013), no. 1, 124–136.
7. O. O. Bezushchak and V. I. Sushchansky, *Групи періодично визначених лінійних перетворень нескінченновимірного векторного простору* [*Groups of periodically defined linear transformations of an infinite-dimensional vector space*], Укр. мат. журн. **67** (2015), no. 10, 1299–1308 = Ukrainian Math. J. **67** (2016), no. 10, 1457–1468.
8. O. Bezushchak, B. Oliynyk, and V. Sushchansky, *Representation of Steinitz's lattice in lattices of substructures of relational structures*, Algebra Discrete Math. **21** (2016), no. 2, 184–201.
9. B. Oliynyk, *The diagonal limits of Hamming spaces*, Algebra Discrete Math. **15** (2013), no. 2, 229–236.
10. F. Blanchard, E. Formenti, and P. Kůrka, *Cellular automata in the Cantor, Besicovitch, and Weyl topological spaces*, Complex Systems **11** (1997), no. 2, 107–123.
11. O. Ore, *Theory of monomial groups*, Trans. Amer. Math. Soc. **51** (1942), no. 1, 15–64.
12. M. Ch. Klin, R. Pöschel, and K. Rosenbaum, *Angewandte Algebra für Mathematiker und Informatiker. Einführung in gruppentheoretisch-kombinatorische Methoden* [*Applied algebra for mathematicians and computer scientists. Introduction to group-theoretical combinatorial methods*], Dtsch. Verl. Wiss., Berlin, 1988.
13. M. Kuzucuoğlu, B. Oliynyk, and V. I. Sushchanskyy, *Homogeneous monomial groups and centralizers*, Comm. Algebra **46** (2018), no. 2, 597–609.

Department of Mathematics, National University of Kyiv-Mohyla Academy, 2 Skovorody St., Kyiv, 04070, Ukraine
Email: oliynyk@ukma.edu.ua

Voronoï's Impact on Modern Science, Book 6, Volume 1
Proceedings of the Sixth International Conference on Analytic Number Theory and Spatial Tessellations

THE RIEMANN ZETA-FUNCTION ON HORIZONTAL SHIFTS OF ITS COMPLEX ZEROS

Teerapat Srichan (*Thailand*) and Jörn Steuding (*Germany*)

Abstract. In this short note we present a different proof of part of a formula recently obtained by Garunkštis and Laurinčikas on the values of the zeta-function at shifts of its complex zeros. Our approach provides a smaller error term.

1. Statement of the main result

Let $s = \sigma + it$ denote a complex variable. Recently, assuming the truth of the Riemann hypothesis, Ramūnas Garunkštis and Antanas Laurinčikas [3] proved for the values of the Riemann zeta-function ζ at $s+i\gamma$ the following discrete mean-square formula: *for $\sigma > \frac{1}{2}$ and $|t| \le T^A$ with arbitrary positive A,*

$$\sum_{0<\gamma\le T} |\zeta(s+i\gamma)|^2 = \zeta(2\sigma)\frac{T}{2\pi}\log\frac{T}{2\pi e} + \zeta(2\sigma)\mathrm{Re}\,\frac{\zeta'}{\zeta}(s+\tfrac{1}{2})\frac{T}{\pi} + O(T^{1-\eta}), \quad (1)$$

where η is some positive real number (not further specified) *and the implicit constant in the error term depends on σ and A.* Notice that the Riemann hypothesis (RH) claims that all non-real zeros $\rho = \beta + i\gamma$ have real part $\beta = \frac{1}{2}$. In this note we shall specify the error term for the case of real s. The method of proof in [3] relies on a certain approximation of ζ by a rather short Dirichlet polynomial (thanks to the assumption of RH) whereas our approach relies on contour integration. Our main result is unconditional and deals with

$$\mathcal{M}(\alpha) := \sum_{0<\gamma<T} \zeta(\alpha - \tfrac{1}{2} + \rho)\zeta(\alpha + \tfrac{1}{2} - \rho),$$

where the summation is over all nontrivial zeros $\rho = \beta + i\gamma$ satisfying $0 < \gamma < T$ and α is assumed to be real.

Received July 31, 2018; revised November 12, 2018.
2010 *Mathematics Subject Classification.* 11M06, 11M26.
Key words and phrases. Riemann zeta-function, non-trivial zeros.

Theorem 1. *For every fixed $\alpha \in (\frac{1}{2}, 1)$, as $T \to \infty$,*

$$\mathcal{M}(\alpha) = \zeta(2\alpha)\frac{T}{2\pi}\log\frac{T}{2\pi e} + \zeta(2\alpha)\frac{\zeta'}{\zeta}(\tfrac{1}{2}+\alpha)\frac{T}{\pi} + O\left(T^{\frac{1}{2}}(\log T)^4 + T^{\frac{3}{2}-\alpha}\right); \quad (2)$$

this holds uniformly in α for α from any compact subset of $(\frac{1}{2}, 1)$.

If the Riemann hypothesis is true, then

$$\mathcal{M}(\alpha) = \sum_{0<\gamma\le T} |\zeta(\alpha + i\gamma)|^2$$

and formula (1) of Garunkštis and Laurinčikas is recovered for $t = 0$ with an explicit error term from (2). Notice that Ivić [7] proved unconditionally that

$$\sum_{0<\gamma<T} |\zeta(\tfrac{1}{2} + i\gamma)|^2 \ll T(\log T)^2 \log\log\log T.$$

Assuming the truth of the Riemann hypothesis, Steven Gonek [4] showed for real δ satisfying $|\delta| \le \frac{\log T}{4\pi}$ that

$$\sum_{0<\gamma<T} \left|\zeta\left(\frac{1}{2} + i\left(\gamma + \frac{2\pi\delta}{\log\frac{T}{2\pi}}\right)\right)\right|^2 = \left(1 - \left(\frac{\sin\pi\delta}{\pi\delta}\right)^2\right)\frac{T}{2\pi}(\log T)^2 + O(T\log T);$$

the error term of this formula had been improved by Akio Fujii [2]. This result is about vertical shifts around the zeros and therefore is related to questions about the pair correlation of zeros, a topic originating from the work [10] of Hugh Montgomery. Our result deals with horizontal shifts, however, it is not uniform in α and thus we cannot conclude any asymptotics by letting $\alpha \to \frac{1}{2}$. To see that one may observe that for $\alpha = \frac{1}{2} + \frac{\delta}{\log T}$ the Laurent expansions of $\zeta(2\alpha)$ and $\frac{\zeta'}{\zeta}(\frac{1}{2}+\alpha)$ would lead to a negative main term for $\delta < 2$ although $\mathcal{M}(\alpha)$ is supposed to be non-negative under assumption of the Riemann hypothesis.

2. The proof

Our proof follows along the lines of the article [1] by Brian Conrey, Amit Ghosh, and Steven Gonek (and in some literature this is called the method of Conrey, Ghosh, and Gonek although the idea of such kind of

contour integration can be traced back to Edmund Landau 1911/12 [9]). By Cauchy's theorem we rewrite the sum $\mathcal{M}(\alpha)$ in terms of an integral,

$$\mathcal{M}(\alpha) = \frac{1}{2\pi i}\int_{\mathcal{R}} \frac{\zeta'}{\zeta}(s)\zeta(\alpha - \tfrac{1}{2} + s)\zeta(\alpha + \tfrac{1}{2} - s)\,\mathrm{d}s,$$

where $\mathcal{R}$ is a counterclockwise oriented rectangular contour with vertices $a+i$, $a+iT$, $1-a+iT$, $1-a+i$, where $a = \alpha + \frac{1}{2} + \mathcal{L}^{-1}$ with $\mathcal{L} := \log \frac{T}{2\pi}$. The latter formula follows easily form the calculus of residues by noting that every zero ρ of ζ of multiplicity $m(\rho)$ appears as simple zero of the logarithmic derivative with residue $m(\rho)$; the pole of ζ (resp. of its logarithmic derivative) lies outside $\mathcal{R}$. The height T of the rectangle here is chosen such that the distance of T to any ordinate γ of a zero is at least $c\mathcal{L}^{-1}$, where c is some positive constant; this follows in a standard way from the Riemann–von Mangoldt formula which provides for the number $N(T)$ of zeros $\rho = \beta + i\gamma$ with $0 < \gamma < T$ the asymptotic value

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi e} + O(\mathcal{L}).$$

For this and further basic knowledge about the zeta-function we refer to Aleksandar Ivić's monograph [6].

We may rewrite the integral over $\mathcal{R}$ as the sum of four integrals over the respective line segments. In view of the choice of T we deduce from the partial fraction expansion of the logarithmic derivative that

$$\frac{\zeta'}{\zeta}(\sigma + it) = \sum_{|t-\gamma|\le 1}\frac{1}{s-\rho} + O(\mathcal{L}) \ll \mathcal{L}^2,$$

valid for $\sigma \in [-1, 2]$ and $t \in [1, T]$. Moreover,

$$\zeta(\sigma + it) \ll \mathcal{L}t^{\mu(\sigma)} \quad \text{with} \quad \mu(\sigma) = \begin{cases} 0 & \text{for } \sigma > 1, \\ \frac{1}{2}(1-\sigma) & \text{for } 0 \le \sigma \le 1, \\ \frac{1}{2} - \sigma & \text{for } \sigma < 0. \end{cases}$$

Hence, the integrals over the horizontal paths contribute

$$\begin{aligned} &\ll 1 + \int_{1-a}^{a}\left|\frac{\zeta'}{\zeta}(\sigma + iT)\zeta(\alpha - \tfrac{1}{2} + \sigma + iT)\zeta(\alpha + \tfrac{1}{2} - \sigma + iT)\right|\,\mathrm{d}\sigma \\ &\ll 1 + \left\{\int_{-\frac{1}{2}-\alpha-\mathcal{L}^{-1}}^{\alpha-\frac{1}{2}} + \int_{\alpha-\frac{1}{2}}^{\frac{3}{2}-\alpha} + \int_{\frac{3}{2}-\alpha}^{\frac{1}{2}+\alpha+\mathcal{L}^{-1}}\right\}''\,\mathrm{d}\sigma \\ &\ll 1 + \ T^{\frac{1}{2}}\mathcal{L}^4 + T^{1-\alpha}\mathcal{L}^2 + T^{\frac{1}{2}}\mathcal{L}^4 \\ &\ll T^{\frac{1}{2}}\mathcal{L}^4. \end{aligned}$$

Next we shall consider the integrals over the vertical paths; they contribute to the main term. For this purpose we recall the functional equation for ζ, i.e.,

$$\zeta(s) = \Delta(s)\zeta(1-s) \tag{3}$$

with

$$\Delta(s) = \pi^{s-\frac{1}{2}}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})} = \left(\frac{t}{2\pi}\right)^{\frac{1}{2}-\sigma-it} \exp(i(t+\tfrac{\pi}{4}))(1+O(t^{-1})). \tag{4}$$

Differentiation of the functional equation yields for the integral on the left vertical path of $\mathcal{R}$

$$\begin{aligned}\frac{1}{2\pi i}\int_{1-a+iT}^{1-a+i} &\frac{\zeta'}{\zeta}(s)\zeta(\alpha-\tfrac{1}{2}+s)\zeta(\alpha+\tfrac{1}{2}-s)\,\mathrm{d}s\\ &= \frac{1}{2\pi i}\int_{1-a+iT}^{1-a+i}\left(\frac{\zeta'}{\zeta}(1-s)-\frac{\Delta'}{\Delta}(s)\right)\times\\ &\qquad\times\Delta(\alpha-\tfrac{1}{2}+s)\zeta(\tfrac{3}{2}-\alpha-s)\zeta(\alpha+\tfrac{1}{2}-s)\,\mathrm{d}s\\ &= \mathcal{J}_1-\mathcal{J}_2,\end{aligned} \tag{5}$$

say. We observe that the ζ-terms here are all representable as absolutely convergent Dirichlet series on the line of integration, namely $\zeta(s)=\sum_{n\ge1}n^{-s}$ and $\frac{\zeta'}{\zeta}(s)=-\sum_{m\ge2}\Lambda(m)m^{-s}$ with the von Mangoldt Λ-function defined by $\Lambda(m)=\log p$ whenever $m=p^\nu$ is the power of a prime number, and $\Lambda(m)=0$ otherwise.

Hence, substituting (4) leads to

$$\mathcal{J}_1 = -\sum_{m\ge2}\frac{\Lambda(m)}{m^a}\sum_{n\ge1}\frac{n^{\alpha-\frac{1}{2}}}{n^a}\sum_{\ell\ge1}\frac{\ell^{\frac{1}{2}-\alpha}}{\ell^a}\exp(-\tfrac{\pi i}{4})\,\overline{\mathcal{I}},$$

where $\overline{\mathcal{I}}$ is the conjugate of the integral

$$\begin{aligned}\mathcal{I} &:= \frac{1}{2\pi}\int_1^T \Delta(\alpha-a+\tfrac{1}{2}+it)(mn\ell)^{-it}\,\mathrm{d}t\\ &= \frac{1}{2\pi}\int_1^T\left(\frac{t}{2\pi}\right)^{a-\alpha}\exp\left(it\log\frac{t}{2\pi emn\ell}\right)(1+O(t^{-1}))\,\mathrm{d}t.\end{aligned}$$

Here we proceed with an old lemma due to Gonek [4]: *for sufficiently large A, uniformly in b,*

$$\begin{aligned}&\frac{1}{2\pi}\int_A^B\left(\frac{t}{2\pi}\right)^{b-\frac{1}{2}}\exp\left(it\log\frac{t}{er}\right)\,\mathrm{d}t\\ &\quad=\begin{cases}\left(\frac{r}{2\pi}\right)^b\exp\left(\frac{\pi i}{4}-ir\right)+O(A^{b-\frac{1}{2}}) & \text{if } A\le r\le B\le 2A,\\ O(A^{b-\frac{1}{2}}) & \text{if } r<A \text{ or } r>B.\end{cases}\end{aligned} \tag{6}$$

Using this we get

$$\begin{aligned}
&\frac{1}{2\pi}\int_1^T \left(\frac{t}{2\pi}\right)^{a-\alpha} \exp\left(it\log\frac{t}{2\pi er}\right)\,\mathrm{d}t \\
&\quad= \sum_{1\le j\le \mathcal{L}/\log 2} \frac{1}{2\pi}\int_{2^{-j}T}^{2^{1-j}T} \left(\frac{t}{2\pi}\right)^{a-\alpha} \exp\left(it\log\frac{t}{2\pi er}\right)\,\mathrm{d}t \\
&\quad= \sum_{1\le j\le \mathcal{L}/\log 2} \left\{ \begin{array}{ll} \left(\frac{r}{2\pi}\right)^{a+\frac12\exp\left(\frac{\pi i}{4}-ir\right)-\alpha} + O((2^{-j}T)^{a-\alpha}) & \\ \qquad\qquad\qquad\qquad \text{if } 2^{-j}T\le r\le 2^{1-j}T, & \\ O((2^{-j}T)^{a-\alpha}) \quad \text{if } r<2^{-j}T \text{ or } r>2^{1-j}T & \end{array} \right\} \\
&\quad= \begin{cases} \left(\frac{r}{2\pi}\right)^{a+\frac12-\alpha}\exp\left(\frac{\pi i}{4}-ir\right)+O(T^{a-\alpha}) & \text{if } 1\le r\le T, \\ O(T^{a-\alpha}) & \text{if } r<1 \text{ or } r>T. \end{cases}
\end{aligned}$$

Applying this with $r=mn\ell$ yields

$$\mathcal{J}_1 = -\sum_{mn\ell\le\frac{T}{2\pi}} \Lambda(m)m^{\frac12-\alpha}n^{1-2\alpha} + O(T^{\frac12})$$

(since the series over the j's converges and $T^{\mathcal{L}^{-1}}\ll 1$). Here the summation is over all integers $m\ge 2$ and n, $\ell\ge 1$ satisfying the condition $mn\ell\le\frac{T}{2\pi}$. It is straightforward to evaluate the sum as follows:

$$\begin{aligned}
\sum_{mn\ell\le\frac{T}{2\pi}} \Lambda(m)m^{\frac12-\alpha}n^{1-2\alpha} &= \sum_{mn\le\frac{T}{2\pi}} \Lambda(m)m^{\frac12-\alpha}n^{1-2\alpha} \sum_{\ell\le\frac{T}{2\pi mn}} 1 \\
&= \sum_{m\le\frac{T}{2\pi}} \Lambda(m)m^{-\frac12-\alpha} \sum_{n\le\frac{T}{2\pi m}} n^{-2\alpha}\left(\frac{T}{2\pi}+O(1)\right).
\end{aligned}$$

In view of $\alpha>\frac12$ both sums converge as $T\to\infty$. Since

$$\sum_{n\le\frac{T}{2\pi m}} n^{-2\alpha} = \sum_{n\ge 1} n^{-2\alpha} - \sum_{n>\frac{T}{2\pi m}} n^{-2\alpha} = \zeta(2\alpha) + O\left(\left(\frac{T}{m}\right)^{1-2\alpha}\right)$$

and

$$\begin{aligned}
\sum_{2\le m\le\frac{T}{2\pi}} \Lambda(m)m^{-\frac12-\alpha} &= \sum_{m\ge 2} \Lambda(m)m^{-\frac12-\alpha} - \sum_{m>\frac{T}{2\pi}} \Lambda(m)m^{-\frac12-\alpha} \\
&= -\frac{\zeta'}{\zeta}(\tfrac12+\alpha) + O\left(T^{\frac12-\alpha}\right),
\end{aligned}$$

we thus get

$$\mathcal{J}_1 = \frac{\zeta'}{\zeta}(\tfrac12+\alpha)\zeta(2\alpha)\frac{T}{2\pi} + O\left(T^{\frac32-\alpha}\right). \tag{7}$$

Concerning $\mathcal{J}_2$ (see (5) above), we may proceed in a rather similar way. Notice that

$$\frac{\Delta'}{\Delta}(s) = -\log\frac{t}{2\pi} + O(t^{-1})$$

for $t \geq 1$. Thus, we find

$$\begin{aligned}\mathcal{J}_2 &= -\frac{1}{2\pi i}\int_{1-a+i}^{1-a+iT}\left(\log\frac{t}{2\pi} + O(t^{-1})\right)\times \\ &\qquad \times \Delta(\alpha-\tfrac{1}{2}+s)\zeta(\tfrac{3}{2}-\alpha-s)\zeta(\tfrac{1}{2}+\alpha-s)\,\mathrm{d}s \\ &= -\int_1^T \left(\log\frac{t}{2\pi} + O(t^{-1})\right)\mathrm{d}\,\mathcal{H}(t), \end{aligned} \tag{8}$$

where

$$\mathcal{H}(\tau) := \frac{1}{2\pi i}\int_{1-a+i}^{1-a+i\tau}\Delta(\alpha-\tfrac{1}{2}+s)\zeta(\tfrac{3}{2}-\alpha-s)\zeta(\tfrac{1}{2}+\alpha-s)\,\mathrm{d}s.$$

The latter integral can be computed by means of (6) as

$$\begin{aligned}\mathcal{H}(\tau) &= \sum_{n\geq 1}\frac{n^{\alpha-\frac{1}{2}}}{n^a}\sum_{\ell\geq 1}\frac{\ell^{\frac{1}{2}-\alpha}}{\ell^a}\,\exp(-\tfrac{\pi}{4})\times \\ &\qquad \times\frac{1}{2\pi}\int_1^\tau\left(\frac{t}{2\pi}\right)^{a-\alpha}\exp\left(it\log\frac{t}{2\pi en\ell}\right)(1+O(t^{-1}))\,\mathrm{d}t \\ &= \sum_{n\ell\leq\frac{\tau}{2\pi}}\ell^{1-2\alpha} + O(\tau^{\frac{1}{2}}) \\ &= \frac{\tau}{2\pi}\zeta(2\alpha) + O(\tau^{\frac{1}{2}}+\tau^{2-2\alpha}).\end{aligned}$$

This gives via (8)

$$\mathcal{J}_2 = -\zeta(2\alpha)\frac{T}{2\pi}\log\frac{T}{2\pi e} + O(T^{\frac{1}{2}}+T^{2-2\alpha}). \tag{9}$$

Finally, we observe that the integral on the right vertical path is an old friend, namely

$$\frac{1}{2\pi i}\int_{a+i}^{a+iT}\frac{\zeta'}{\zeta}(s)\zeta(\alpha-\tfrac{1}{2}+s)\zeta(\alpha+\tfrac{1}{2}-s)\,\mathrm{d}s = \overline{\mathcal{J}}_1,$$

as follows from (5). Hence, $\mathcal{M}(\alpha) = 2\,\mathrm{Re}\,\mathcal{J}_1 - \mathcal{J}_2$ and the first formula (2) of the theorem follows from (7) and (9).

It should be technical but not difficult to prove the full result of [3] along this method, of course, conditional on the Riemann hypothesis.

3. A final word

On the first glance, Theorem 1 has nothing to do with the mathematics of Georgy Voronoï, however, the disciplines of mathematics are widespread. In fact, works of Hans Hamburger [5], Nikolai Koshliakov [8], Fritz Oberhettinger and Kusum Soni [11] showed that Voronoï's famous sum formula is equivalent to the functional equation of the Riemann zeta-function (3) which appeared in our reasoning as a main ingredient.

References

1. J. B. Conrey, A. Ghosh, and S. M. Gonek, *Simple zeros of the zeta function of a quadratic number field.* I, Invent. Math. **86** (1986), 563–576.
2. A. Fujii, *On a mean value theorem in the theory of the Riemann zeta function*, Comment. Math. Univ. St. Pauli **44** (1995), 59–67.
3. R. Garunkštis and A. Laurinčikas, *Discrete mean square of the Riemann zeta-function over imaginary parts of its zeros*, Period. Math. Hungar. **76** (2018), 217–228.
4. S. M. Gonek, *Mean values of the Riemann zeta-function and its derivatives*, Invent. Math. **75** (1984), 123–141.
5. H. Hamburger, *Über einige Beziehungen, die mit der Funktionalgleichung der Riemannschen ζ-Funktion äquivalent sind*, Math. Ann. **85** (1922), 129–140.
6. A. Ivić, *The theory of the Riemann zeta-function with applications*, John Wiley & Sons, New York, 1985.
7. A. Ivić, *On sums of squares of the Riemann zeta-function on the critical line*, Proc. Session Anal. Number Theory Diophantine Equ., Bonner Math. Schriften, vol. 360, Univ. Bonn, Bonn, 2003, 17 pp.
8. N. S. Koshliakov, *On Voronoï's sum-formula*, Messenger **58** (1928), 30–32.
9. E. Landau, *Über die Nullstellen der Zetafunktion*, Math. Ann. **71** (1912), 548–564.
10. H. L. Montgomery, *The pair correlation of zeros of the zeta function*, Analytic Number Theory (Proc. Sympos. Pure Math., vol. 24, St. Louis, MO, 1972), Amer. Math. Soc., Providence, RI, 1973, pp. 181–193.
11. F. Oberhettinger and K. L. Soni, *On some relations which are equivalent to functional equations involving the Riemann zeta-function*, Math. Z. **127** (1972), 17–34.

Department of Mathematics, Faculty of Science, Kasetsart University, Bangkok, 10900, Thailand
Email: fscitrp@ku.ac.th

Department of Mathematics, Würzburg University, 40 Emil Fischer-Str., Würzburg, 97 074, Germany
Email: steuding@mathematik.uni-wuerzburg.de

MULTIVARIABLE AVERAGES OF DIVISOR FUNCTIONS

László Tóth (*Hungary*) and Wenguang Zhai (*China*)

We discuss asymptotic formulas for the sums

$$\sum_{n_1,\dots,n_r\le x} f(n_1\cdots n_r) \quad \text{and} \quad \sum_{n_1,\dots,n_r\le x} f([n_1,\dots,n_r]),$$

where $r\ge 2$ is a fixed integer, $[n_1,\dots,n_r]$ stands for the least common multiple of the integers $n_1,\dots,n_r$ and f is one of the divisor functions $\tau_{1,k}(n)=\sum_{d^k\mid n}1$ $(k\ge 1)$, $\tau^{(e)}(n)$, representing the number of exponential divisors of n, and $\tau^*(n)=2^{\omega(n)}$.

For example, we show that if $k,r\ge 2$, then

$$\sum_{n_1,\dots,n_r\le x}\tau_{1,k}(n_1\cdots n_r)=A_{r,k}x^r+B_{r,k}x^{r-1+1/k}+O(x^{r-1+\theta_k+\varepsilon}),$$

$$\sum_{n_1,\dots,n_r\le x}\tau_{1,k}([n_1,\dots,n_r])=C_{r,k}x^r+D_{r,k}x^{r-1+1/k}+O(x^{r-1+\theta_k+\varepsilon}),$$

for every $\varepsilon>0$, where θ_k is the exponent in the error term of the classical formula

$$\sum_{n\le x}\tau_{1,k}(n)=\zeta(k)x+\zeta(1/k)x^{1/k}+O(x^{\theta_k+\varepsilon}),$$

and $A_{r,k}$, $B_{r,k}$, $C_{r,k}$, $D_{r,k}$ are computable constants.

Our formulas refine and generalize a result of A. V. Lelechenko (2014). A new generalization of the Busche–Ramanujan identity is also pointed out.

The paper in full form with all the proofs of the statements is already published. See L. Tóth and W. Zhai, *On multivariable averages of divisor functions*, J. Number Theory **192** (2018), 251–269.

Department of Mathematics, University of Pécs, 6 Ifjúság útja, Pécs, H-7624, Hungary
Email: ltoth@gamma.ttk.pte.hu

Department of Mathematics, China University of Mining and Technology, Beijing, 100083, China

Received August 24, 2018; revised November 5, 2018.
2010 *Mathematics Subject Classification.* 11A25, 11N37.
Key words and phrases. Divisor function, least common multiple, multiple Dirichlet series, average order, error term, Busche–Ramanujan identity.

ON COMBINATORIAL IDENTITIES FOR JACOBSTHAL POLYNOMIALS

Taras Goy (*Ukraine*)

Abstract. We consider determinants for some families of Toeplitz–Hessenberg matrices, whose entries are the Jacobsthal polynomials. These formulas may also be rewritten as identities involving sums of products of Jacobsthal polynomials (with sequential, odd, or even subscripts) and multinomial coefficients.

Introduction

Many numbers and polynomial sequences can be defined by second-order recurrence relations such as Fibonacci numbers, Lucas numbers, Pell numbers, Jacobsthal numbers, Chebyshev polynomials, etc. These numbers and polynomials play a fundamental role in mathematics and have numerous important applications in combinatorics, number theory, and numerical analysis (see, e.g., [3, 10, 11] and references therein).

In particular, the *Jacobsthal sequence* $\{J_n\}_{n\geq 0}$ is defined by the recurrence relation

$$J_n = J_{n-1} + 2J_{n-2},$$

with initial conditions $J_0 = 0$, $J_1 = 1$ (sequence A001045 in the On-Line Encyclopedia of Integer Sequences [15]).

A natural extension of the Jacobsthal numbers is given by the *Jacobsthal polynomials* $\{j_n(x)\}_{n\geq 0}$, which are introduced by Horadam in [9] and defined as follows:

$$j_n(x) = j_{n-1}(x) + 2j_{n-2}(x), \quad n \geq 2, \tag{1}$$

with $j_0(x) = 0$, $j_1(x) = 1$. The explicit closed-form expression for $j_n(x)$ is

$$j_n(x) = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-1-i}{i} (2x)^i, \quad n \geq 0,$$

where $\lfloor \alpha \rfloor$ is the floor function.

Received July 15, 2018; revised October 28, 2018.
2010 *Mathematics Subject Classification.* Primary 11B37; Secondary 15A15.
Key words and phrases. Jacobsthal polynomial, Jacobsthal sequence, Fibonacci sequence, Toeplitz–Hessenberg matrix, multinomial coefficient.

Note that $j_n(1) = J_n$ and $j_n(1/2) = F_n$ are the nth Fibonacci numbers (sequence A000045 in [15]).

The first few Jacobsthal polynomials are

$$\begin{aligned} j_0(x) &= 0, & j_1(x) &= j_2(x) = 1, \\ j_3(x) &= 2x + 1, & j_4(x) &= 4x + 1, \\ j_5(x) &= 4x^2 + 6x + 1, & j_6(x) &= 12x^2 + 8x + 1, \\ j_7(x) &= 8x^3 + 24x^2 + 10x + 1, & j_8(x) &= 32x^3 + 40x^2 + 12x + 1. \end{aligned}$$

The Jacobsthal polynomials have many applications (see, e.g., papers [1, 4, 14]). Many interesting properties of Jacobsthal polynomials and their generalizations are studied in [2, 3, 9, 16].

Our purpose is to study the Jacobsthal polynomials and to investigate some families of Toeplitz–Hessenberg matrices, whose entries are Jacobsthal polynomials with successive, odd, or even subscripts. As a result, we obtain some new identities with multinomial coefficients for these polynomials.

1. Toeplitz–Hessenberg matrices and determinants

A lower *Toeplitz–Hessenberg matrix* is a square matrix of the form

$$M_n(a_0; a_1, \ldots, a_n) = \begin{bmatrix} a_1 & a_0 & 0 & \cdots & 0 & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & a_0 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 \end{bmatrix},$$

where $a_0 \neq 0$ and $a_i \neq 0$ for at least one $i > 0$.

Expanding the determinant $\det(M_n)$ repeatedly along the last row, we obtain the recurrence relation

$$\det(M_n) = \sum_{i=1}^{n} (-a_0)^{i-1} a_i \det(M_{n-i}), \tag{2}$$

where, by definition, $\det(M_0) = 1$.

It is known that the determinant $\det(M_n)$ can be evaluated, by using the *Trudi formula* as follows:

$$\det(M_n) = \sum_{s_1+2s_2+\cdots+ns_n=n} p_n(s)(-a_0)^{n-|s|} a_1^{s_1} \cdots a_n^{s_n}, \tag{3}$$

where $|s| = s_1 + \cdots + s_n$, $p_n(s) = \binom{|s|}{s_1,\ldots,s_n}$ is the multinomial coefficient, and $n = s_1 + 2s_2 + \cdots + ns_n$ is the partition of a positive integer n, where each positive integer i appears s_i times [12, 13].

Many combinatorial identities involving sums over the partitions of integers can be generated in this way. Some of these identities presented in [5, 6, 7] and Section 3 of this paper.

2. Toeplitz–Hessenberg determinants, whose entries are Jacobsthal polynomials

In this section, we provide determinant formulas for Toeplitz–Hessenberg matrices, whose triangular entries are Jacobsthal polynomials with successive, odd, or even subscripts.

We investigate particular cases of Toeplitz–Hessenberg matrices, where $a_0 = \pm 1$. For simplicity of notation, we write $\det(\pm 1; a_1, \dots, a_n)$ in place of $\det\big(M_n(\pm 1; a_1, \dots, a_n)\big)$.

Theorem 1. *Let $n \geq 1$, unless otherwise stated. Then*

$$\det(1; j_0(x), \dots, j_{n-1}(x)) = \frac{1}{\sqrt{8x-3}}\left(\left(\frac{-1-\sqrt{8x-3}}{2}\right)^{n-1} - \left(\frac{-1+\sqrt{8x-3}}{2}\right)^{n-1}\right);$$

$$\det(-1; j_0(x), \dots, j_{n-1}(x)) = \frac{1}{\sqrt{8x+5}}\left(\left(\frac{1+\sqrt{8x+5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{8x+5}}{2}\right)^{n-1}\right);$$

$$\det(1; j_0(x), \dots, j_{2n-2}(x)) = \frac{1}{\sqrt{8x-3}}\left(\left(\frac{-4x-1-\sqrt{8x-3}}{2}\right)^{n-1} - \left(\frac{-4x-1+\sqrt{8x-3}}{2}\right)^{n-1}\right);$$

$$\det(-1; j_0(x), \dots, j_{2n-2}(x)) = \frac{1}{\sqrt{8x+5}}\left(\left(\frac{4x+1+\sqrt{8x+5}}{2}\right)^{n-1} - \left(\frac{4x+1-\sqrt{8x+5}}{2}\right)^{n-1}\right);$$

$$\det(1; j_1(x), \dots, j_n(x)) = 2^{\frac{n-3}{2}} x^{\frac{n-1}{2}}\big(1-(-1)^n\big); \tag{4}$$

$$\det(-1; j_1(x), \dots, j_n(x)) = \frac{\big(1+\sqrt{2x+1}\big)^n - \big(1-\sqrt{2x+1}\big)^n}{2\sqrt{2x+1}};$$

$$\det(1; j_1(x), \dots, j_{2n-1}(x)) = \frac{\big(-2x-\sqrt{2x}\big)^{n-1} + \big(-2x+\sqrt{2x}\big)^{n-1}}{2};$$

$$\det(1; j_2(x), \dots, j_{n+1}(x)) = 0, \qquad n \geq 3;$$

$$\det(-1; j_2(x), \dots, j_{n+1}(x)) = \frac{\big(1+\sqrt{4x+1}\big)^{n+1} - \big(1-\sqrt{4x+1}\big)^{n+1}}{4\sqrt{4x+1}};$$

$$\det(1; j_2(x), \dots, j_{2n}(x)) = n(-2x)^{n-1};$$

$$\det(-1; j_2(x), \dots, j_{2n}(x)) = \frac{\big(2x+1+\sqrt{4x+1}\big)^n - \big(2x+1-\sqrt{4x+1}\big)^n}{2\sqrt{4x+1}};$$

$$\det(1; j_3(x), \dots, j_{n+2}(x)) = (2x)^n, \qquad n \geq 2;$$

$$\det(-1; j_3(x), \dots, j_{2n+1}(x)) = 2^{n-1} \sum_{k=0}^{n} \sum_{i=0}^{n} \binom{n-i}{k} \binom{n+k}{i} x^{n-k};$$

$$\det(1; j_3(x), \dots, j_{2n+1}(x)) = (-2x)^{n-1}, \qquad n \geq 2;$$

$$\det(1; j_4(x), \dots, j_{n+3}(x)) = (2x)^{n-1}\big(2x(n+1)+1\big);$$

$$\det(1; j_4(x), \dots, j_{2n+2}(x)) = 0, \qquad n \geq 3.$$

Proof. We will prove formula (4), by using induction on n. The other proofs follow similarly, so we omit them for the sake of brevity.

Clearly, formula (4) works, when $n = 1$ and $n = 2$. Suppose it is true for all $k \leq n-1$, where $n \geq 3$.

Let $D_n = \det(1; j_1(x), \dots, j_n(x))$. Using recurrence relations (2) and relation (1), we obtain

$$\begin{aligned}
D_n &= \sum_{s=1}^{n} (-1)^{s-1} j_s(x) D_{n-s} \\
&= j_1(x) D_{n-1} + \sum_{s=2}^{n} (-1)^{s-1} \big(j_{s-1}(x) + 2x j_{s-2}(x)\big) D_{n-s} \\
&= D_{n-1} + \sum_{s=2}^{n} (-1)^{s-1} j_{s-1}(x) D_{n-s} + 2x \sum_{s=2}^{n} (-1)^{s-1} j_{s-2}(x) D_{n-s} \\
&= D_{n-1} + \sum_{s=1}^{n-1} (-1)^{s} j_s(x) D_{n-1-s} + 2x \sum_{s=0}^{n-2} (-1)^{s+1} j_s(x) D_{n-2-s} \\
&= 2x D_{n-2}.
\end{aligned}$$

Using the induction hypothesis, we have

$$\begin{aligned}
D_n &= 2x \cdot 2^{\frac{n-5}{2}} x^{\frac{n-3}{2}} \big(1 - (-1)^{n-2}\big) \\
&= 2^{\frac{n-3}{2}} x^{\frac{n-1}{2}} \big(1 - (-1)^n\big).
\end{aligned}$$

Consequently, formula (4) is true for n. Therefore, by induction, the formula works for all integers $n > 0$. □

3. New formulas with multinomial coefficients for Jacobsthal polynomials

Next, we focus on a multinomial extension of Theorem 1. Trudi's formula (3) with Theorem 1 yield the following result.

Theorem 2. *Let $n \geq 1$, unless otherwise stated. Then*

$$\sum_{\sigma_n=n} (-1)^{|s|} p_n(s) j_0^{s_1}(x) \cdots j_{n-1}^{s_n}(x) = \frac{1}{\sqrt{8x-3}} \left(\left(\frac{1-\sqrt{8x-3}}{2} \right)^{n-1} - \left(\frac{1+\sqrt{8x-3}}{2} \right)^{n-1} \right);$$

$$\sum_{\sigma_n=n} p_n(s) j_0^{s_1}(x) \cdots j_{n-1}^{s_n}(x) = \frac{1}{\sqrt{8x+5}} \left(\left(\frac{1+\sqrt{8x+5}}{2} \right)^{n-1} - \left(\frac{1-\sqrt{8x+5}}{2} \right)^{n-1} \right);$$

$$\sum_{\sigma_n=n} (-1)^{|s|} p_n(s) j_0^{s_1}(x) \cdots j_{2n-2}^{s_n}(x) = \frac{1}{\sqrt{8x-3}} \left(\left(\frac{4x+1-\sqrt{8x-3}}{2} \right)^{n-1} - \left(\frac{4x+1+\sqrt{8x-3}}{2} \right)^{n-1} \right);$$

$$\sum_{\sigma_n=n} p_n(s) j_0^{s_1}(x) \cdots j_{2n-2}^{s_n}(x) = \frac{1}{\sqrt{8x+5}} \left(\left(\frac{4x+1+\sqrt{8x+5}}{2} \right)^{n-1} - \left(\frac{4x+1-\sqrt{8x+5}}{2} \right)^{n-1} \right);$$

$$\sum_{\sigma_n=n} (-1)^{|s|} p_n(s) j_1^{s_1}(x) \cdots j_n^{s_n}(x) = (2x)^{\frac{n-3}{2}} x\big((-1)^n - 1\big);$$

$$\sum_{\sigma_n=n} p_n(s) j_1^{s_1}(x) \cdots j_n^{s_n}(x) = \frac{\big(1+\sqrt{2x+1}\big)^n - \big(1-\sqrt{2x+1}\big)^n}{2\sqrt{2x+1}};$$

$$\sum_{\sigma_n=n} (-1)^{|s|} p_n(s) j_1^{s_1}(x) \cdots j_{2n-1}^{s_n}(x) = \frac{\big(2x+\sqrt{2x}\big)^{n-1} + \big(2x-\sqrt{2x}\big)^{n-1}}{-2};$$

$$\sum_{\sigma_n=n} (-1)^{|s|} p_n(s) j_2^{s_1}(x) \cdots j_{n+1}^{s_n}(x) = 0, \qquad n \geq 3;$$

$$\sum_{\sigma_n=n} p_n(s) j_2^{s_1}(x) \cdots j_{n+1}^{s_n}(x) = \frac{\big(1+\sqrt{4x+1}\big)^{n+1} - \big(1-\sqrt{4x+1}\big)^{n+1}}{4\sqrt{4x+1}};$$

$$\sum_{\sigma_n=n}(-1)^{|s|}p_n(s)j_2^{s_1}(x)\cdots j_{2n}^{s_n}(x)=-n(2x)^{n-1};$$

$$\sum_{\sigma_n=n}p_n(s)j_2^{s_1}(x)\cdots j_{2n}^{s_n}(x)=\frac{\left(2x+1+\sqrt{4x+1}\right)^n-\left(2x+1-\sqrt{4x+1}\right)^n}{2\sqrt{4x+1}};$$

$$\sum_{\sigma_n=n}(-1)^{|s|}p_n(s)j_3^{s_1}(x)\cdots j_{n+2}^{s_n}(x)=(-2x)^n,\qquad n\geq 2;$$

$$\sum_{\sigma_n=n}p_n(s)j_3^{s_1}(x)\cdots j_{2n+1}^{s_n}(x)=2^{n-1}\sum_{k=0}^{n}\sum_{i=0}^{n}\binom{n-i}{k}\binom{n+k}{i}x^{n-k};$$

$$\sum_{\sigma_n=n}(-1)^{|s|}p_n(s)j_3^{s_1}(x)\cdots j_{2n+1}^{s_n}(x)=-(2x)^{n-1},\qquad n\geq 2;$$

$$\sum_{\sigma_n=n}(-1)^{|s|}p_n(s)j_4^{s_1}(x)\cdots j_{n+3}^{s_n}(x)=-(-2x)^{n-1}\big(2x(n+1)+1\big);$$

$$\sum_{\sigma_n=n}(-1)^{|s|}p_n(s)j_4^{s_1}(x)\cdots j_{2n+2}^{s_n}(x)=0,\qquad n\geq 3,$$

where $\sigma_n=s_1+2s_2+\cdots+ns_n$, $|s|=s_1+\cdots+s_n$, $p_n(s)=\frac{|s|!}{s_1!\cdots s_n!}$, *and the summation is over integers* $s_i\geq 0$ *satisfying* $\sigma_n=n$.

Corollary. *Since* $j_n(1)=J_n$ *and* $j_n(1/2)=F_n$, *we can obtain multinomial identities for Jacobsthal and Fibonacci numbers. For example, for* $n\geq 1$, *the following formulas hold*:

$$\sum_{\sigma_n=n}(-1)^{|s|}p_n(s)F_0^{s_1}F_1^{s_2}\cdots F_{n-1}^{s_n}=-1;$$

$$\sum_{\sigma_n=n}(-1)^{|s|}p_n(s)J_0^{s_1}J_1^{s_2}\cdots J_{n-1}^{s_n}=-F_n;$$

$$\sum_{\sigma_n=n}(-1)^{|s|}p_n(s)F_1^{s_1}F_2^{s_2}\cdots F_n^{s_n}=\frac{1-(-1)^n}{2};$$

$$\sum_{\sigma_n=n}(-1)^{|s|}p_n(s)J_1^{s_1}J_2^{s_2}\cdots J_n^{s_n}=2^{\frac{n-3}{2}}\left((-1)^n-1\right);$$

$$\sum_{\sigma_n=n}(-1)^{|s|}p_n(s)F_2^{s_1}F_4^{s_2}\cdots F_{2n}^{s_n}=-n;$$

$$\sum_{\sigma_n=n}(-1)^{|s|}p_n(s)J_2^{s_1}J_4^{s_2}\cdots J_{2n}^{s_n}=-n2^{n-1};$$

$$\sum_{\sigma_n=n}(-1)^{|s|}p_n(s)J_4^{s_1}J_5^{s_2}\cdots J_{n+3}^{s_n}=-(-2)^{n-1}(2n+3).$$

References

1. C. Bender, *Fibonacci transmission lines*, Fibonacci Quart. **31** (1993), no. 3, 227–238.
2. P. Catarino and M. L. Morgado, *On generalized Jacobsthal and Jacobsthal–Lucas polynomials*, An. Ştiinţ. Univ. Ovidius Constanţa Ser. Mat. **24** (2016), no. 3, 61–78.
3. G. B. Djordjević and G. V. Milovanović, Special Classes of Polynomials, University of Nis, Leskovac, 2014.
4. L. Fortuna and M. Frasca, *Generating passive systems from recursively defined polynomials*, Int. J. Circuits Syst. Signal Process **6** (2012), no. 2, 179–188.
5. T. Goy, *On Jacobsthal and Jacobsthal–Lucas identities with multinomial coefficients*, Proc. Int. Conf. "Actual Problems of Pure and Applied Mathematics", Almaty, Kazakhstan (2017), 61–64.
6. T. Goy, *On new identities for Mersenne numbers*, Appl. Math. E-Notes **18** (2018), 100–105.
7. T. Goy, *Some families of identities for Padovan numbers*, Proc. Jangjeon Math. Soc. **21** (2018), no. 10, 413–419.
8. A. F. Horadam, *Convolutions for Jacobsthal-type polynomials* Fibonacci Quart., **40** (2002), no. 3, 212–222.
9. A. Horadam, *Jacobsthal representation polynomials*, Fibonacci Quart. **35** (1997), no. 2, 137–148.
10. T. Koshy, *Fibonacci and Lucas Numbers and Applications*, Wiley, New York, 2017.
11. T. Koshy, *Pell and Pell–Lucas Numbers with Applications*, Springer, New York, 2014.
12. M. Merca, *A note on the determinant of a Toeplitz–Hessenberg matrix*, Spec. Matrices **1** (2013), 10–16.
13. T. Muir, *The Theory of Determinants in the Historical Order of Development*, Vol. 3, Dover, New York, 1960.
14. M. H. Reyhani, S. Alikhani, and R. Hasni, *On the independence polynomials of certain molecular graphs*, Digest J. Nanomater. Biostruct. **7** (2012), no. 1, 193–197.
15. N. J. A. Sloane, editor. The On-Line Encyclopedia of Integer Sequences. Available at `https://oeis.org`.
16. A. Tereszkiewicz and I. Wawreniuk, *Generalized Jacobsthal polynomials and special points for them*, Appl. Math. Comput. **268** (2015), 806–814.

Faculty of Mathematics and Computer Science, Vasyl Stefanyk Precarpathian National University, 57 Shevchenko St., Ivano-Frankivs'k, 76018, Ukraine
Email: `tarasgoy@yahoo.com`

DISCRIMINATIVE PRODUCTS AND MULTIPLICATIVE FUNCTIONS

Vichian Laohakosol and Nittiya Pabhapote (*Thailand*)

Abstract. The three concepts of discriminative, partially discriminative and semi-discriminative products of arithmetic functions of one variable are generalized in two directions: first to those of several variables, and second to products of at least two functions in one variable. One further investigation in the one variable case which involves characterizations of multiplicative functions using unitary discriminative products is treated.

Introduction

For $r \in \mathbb{N}$, let $\mathcal{A}_r$ be the set of complex-valued arithmetic functions in r variables, i.e.,

$$\mathcal{A}_r := \{F : \mathbb{N}^r \to \mathbb{C}\}.$$

For F, $G \in \mathcal{A}_r$, their addition and Dirichlet convolution are defined, respectively, by

$$(F+G)(n_1,\dots,n_r) = F(n_1,\dots,n_r) + G(n_1,\dots,n_r),$$
$$(F*G)(n_1,\dots,n_r) = \sum_{d_1|n_1,\dots,d_r|n_r} F(d_1,\dots,d_r)G(n_1/d_1,\dots,n_r/d_r).$$

For $F \in \mathcal{A}_r$, define the i^{th} $(i = 1,\dots,r)$ *component function* $F_i \in \mathcal{A}_1$ of F to be

$$F_i(n) = F(1,\dots,1,n,1,\dots,1),$$

where the right-hand n appears at the i^{th} coordinate. Denote by $\mathbf{1} \in \mathcal{A}_r$ the constant 1 function, i.e.,

$$\mathbf{1}(n_1,\dots,n_r) = 1 \quad (n_1,\dots,n_r \in \mathbb{N}).$$

A function $F \in \mathcal{A}_r \setminus \{0\}$ is said to be *multiplicative* if

$$F(m_1n_1,\dots,m_rn_r) = F(m_1,\dots,m_r)F(n_1,\dots,n_r)$$

Received May 15, 2018; revised November 8, 2018.
2010 *Mathematics Subject Classification.* 11A25.
Key words and phrases. Arithmetic function, multiplicative and completely multiplicative functions, discriminative product.

holds for any $m_1, \ldots, m_r, n_1, \ldots, n_r \in \mathbb{N}$ such that

$$\gcd(m_1 \cdots m_r, n_1 \cdots n_r) = 1,$$

and denote by

$$\mathcal{M}_r := \{F \in \mathcal{A}_r \ ; \ F \text{ is multiplicative}\}.$$

Clearly, if $F \in \mathcal{M}_r$, then $F(1, \ldots, 1) = 1$.

A function $F \in \mathcal{A}_r \setminus \{0\}$ is said to be *firmly multiplicative* if

$$F(m_1 n_1, \ldots, m_r n_r) = F(m_1, \ldots, m_r) F(n_1, \ldots, n_r)$$

holds for any $m_1, \ldots, m_r, n_1, \ldots, n_r \in \mathbb{N}$ such that

$$\gcd(m_1, n_1) = \cdots = \gcd(m_r, n_r) = 1,$$

and denote by

$$\mathcal{F}_r := \{F \in \mathcal{A}_r \ ; \ F \text{ is firmly multiplicative}\}.$$

A function $F \in \mathcal{A}_r \setminus \{0\}$ is said to be *completely multiplicative* if

$$F(m_1 n_1, \ldots, m_r n_r) = F(m_1, \ldots, m_r) F(n_1, \ldots, n_r)$$

holds for any $m_1, \ldots, m_r, n_1, \ldots, n_r \in \mathbb{N}$, and denote by

$$\mathcal{C}_r := \{F \in \mathcal{A}_r \ ; \ F \text{ is completely multiplicative}\}.$$

It is not difficult to see that $\mathcal{C}_r \subset \mathcal{F}_r \subset \mathcal{M}_r$ and the two inclusions are strict. Note that there is no universal agreement regarding multiplicative functions, e.g., in [1], the term 'multiplicative' is used for 'firmly multiplicative'.

In the 1-variable case, the Souriau–Hsu–Möbius function [15, 2] is defined, for $\alpha \in \mathbb{C}$, by

$$\mu_\alpha(n) = \prod_{p \mid n} \binom{\alpha}{\nu_p(n)} (-1)^{\nu_p(n)},$$

where $n = \prod p^{\nu_p(n)}$ denotes the unique prime factorization of $n \in \mathbb{N}$, $\nu_p(n)$ being the largest exponent of the prime p that divides n; when $\alpha = 1$, this corresponds to the classical Möbius function $\mu_1 := \mu$. The r-variable Souriau–Hsu–Möbius function is defined by

$$M_\alpha(n_1, \ldots, n_r) = \mu_\alpha(n_1) \cdots \mu_\alpha(n_r).$$

Let $F \in \mathcal{A}_r$. We say that F is *variable-separated*, denoted by $F \in V_{\text{sep}}$, if F can be written as

$$F(n_1, \ldots, n_r) = f_1(n_1) f_2(n_2) \cdots f_r(n_r), \tag{1}$$

where $f_i \in \mathcal{A}_1$ and

$$f_i(1) = f_j(1) \in \mathbb{R} \setminus \{0\} \quad (1 \le i, j \le r).$$

Variable-separated arithmetic functions are plentiful, because most well-known classical functions such as the 1-constant function $\mathbf{1}$, the generalized Möbius function M_α, the convolution identity function I, all firmly multiplicative functions [17], are variable separated.

For $F \in V_{\text{sep}}$ as given in (1), if $F \in \mathcal{M}_r$, then $f_1(1) = \cdots = f_r(1) = 1$.

Basic properties. The following basic properties are easily checked, cf. [17].

- $(\mathcal{A}_r, +, *)$ is commutative ring with identity I, where

$$I(n_1 \ldots, n_r) = \begin{cases} 1 & \text{if } n_1 = \cdots = n_r = 1, \\ 0 & \text{if } n_1 \cdots n_r > 1; \end{cases}$$

  note that

$$I(n_1, \ldots, n_r) = \delta(n_1) \cdots \delta(n_r),$$

  where $\delta \in \mathcal{A}_1$ is the 1-dimensional identity function with respect to the Dirichlet convolution, i.e.,

$$\delta(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1; \end{cases}$$

- $F \in \mathcal{A}_r$ has a uniquely determined inverse with respect to the convolution, denoted by F^{-1}, if and only if $F(1, \ldots, 1) \neq 0$;
- the inverse of $\mathbf{1}$ is the Möbius function M_1, where

$$M_1(n_1, \ldots, n_r) = \mu(n_1) \cdots \mu(n_r),$$

  so that $M_1 * \mathbf{1} = \delta$;
- if $F \in \mathcal{M}_r$, then it is determined by the values

$$F(p^{\nu_1}, \ldots, p^{\nu_r}),$$

  where p is prime and $\nu_1, \ldots, \nu_r \in \mathbb{N}_0$. More exactly,

$$F(1, \ldots, 1) = 1$$

  and

$$F(n_1, \ldots, n_r) = \prod_p F(p^{\nu_p(n_1)}, \ldots, p^{\nu_p(n_r)}) \quad (n_1, \ldots, n_r \in \mathbb{N}),$$

  where $n_i = \prod_p p^{\nu_p(n_i)}$ $(1 \leq i \leq r)$ is the prime power factorization of n_i, the products being over the primes p and all but a finite number of the exponents $\nu_p(n_i)$ are zero;

- if $F \in \mathcal{F}_r$, then it is determined by its values at
$$(1, \ldots, 1, p^\nu, 1, \ldots, 1),$$
where p runs through the primes and $\nu \in \mathbb{N} \cup \{0\}$, i.e., if $F \in \mathcal{F}_r$, then
$$F(1, \ldots, 1) = 1$$
and
$$F(n_1, \ldots, n_r) = \prod_p \left(F(p^{\nu_p(n_1)}, 1, \ldots, 1) \cdots F(1, \ldots, 1, p^{\nu_p(n_r)}) \right);$$
- if $F \in \mathcal{C}_r$, then it is determined by its values at
$$(1, \ldots, 1, p, 1, \ldots, 1),$$
where p runs through the primes, i.e., if $F \in \mathcal{C}_r$, then
$$F(1, \ldots, 1) = 1$$
and
$$F(n_1, \ldots, n_r) = \prod_p \left(F(p, 1, \ldots, 1)^{\nu_p(n_1)} \cdots F(1, \ldots, 1, p)^{\nu_p(n_r)} \right).$$

In the case $r = 1$, the distributive property of completely multiplicative function discovered by Lambek [6] (see also [9, Proposition 1.10], [14, Theorem 18] and [16]) asserts that $f \in \mathcal{A}_1$ is completely multiplicative, i.e., $f \in \mathcal{C}_1$, if and only if
$$f(g * h) = fg * fh \tag{2}$$
for every pair $g, h \in \mathcal{A}_1$.

This result is further extended by Langford, [7], using the notion of discriminative products, whose r-variable definitions, extending the ones due to Langford, are as follows: let $G, H \in \mathcal{A}_r$. We say that the product $K := G * H$ is *discriminative* if all its component functions are discriminative, i.e., if the relation
$$K_i(n) = G_i(1)H_i(n) + G_i(n)H_i(1) \quad (i = 1, \ldots, r)$$
holds only when n is prime.

We say that the product $K := G * H$ is *partially discriminative* if all its component functions are partially discriminative, i.e., for every prime power p^j $(j \in \mathbb{N})$, the equation
$$K_i(p^j) = G_i(1)H_i(p^j) + G_i(p^j)H_i(1) \quad (i = 1, \ldots, r)$$
implies that $j = 1$.

Closely related is the concept of semi-discriminative product, whose definition in the 1-variable case was introduced by Haukkanen, [4]. For

$G, H \in \mathcal{A}_r$, the product $K := G * H$ is said to be *semi-discriminative* if all its component functions are semi-discriminative, i.e., the relation

$$K_i(n) = G_i(1)H_i(n) + G_i(n)H_i(1) \quad (i = 1, \ldots, r)$$

holds only when $n = 1$ or n is a prime.

Proposition 1. *Let $f \in \mathcal{A}_1$.*

I ([7]) *Assume that $f(1) \neq 0$. Then $f \in \mathcal{C}_1$ if and only if it distributes over some discriminative product.*

II ([7]) *Assume that $f \in \mathcal{M}_1$. Then $f \in \mathcal{C}_1$ if and only if it distributes over some partially discriminative product.*

III ([4]) *Assume that $f(1) = 1$. Then $f \in \mathcal{C}_1$ if and only if it distributes over a semi-discriminative product.*

Fixing in (2), $h = \mu_\alpha \in \mathcal{A}_1$, the Souriau–Hsu–Möbius function, the following results showing that such distributive property (2) yields a necessary and/or sufficient condition for complete multiplicativity were established in [10] using the concepts of discriminative, partially discriminative products and inverse power functions.

Proposition 2. *Let $f, g \in \mathcal{A}_1$ with $g(1) \neq 0$, and let $\alpha \in \mathbb{R} \setminus \{0\}$.*

A. *If f is completely multiplicative, then*

$$f\,(g * \mu_\alpha) = fg * f\mu_\alpha = fg * f^{-\alpha}.$$

B. *Assume that*

$$f\,(g * \mu_\alpha) = fg * f\mu_\alpha. \tag{3}$$

B1. *If $f(1) \neq 0$ and*

$$(g * \mu_\alpha)(n) - g(1)\mu_\alpha(n) - g(n) \neq 0 \tag{4}$$

for all $n \in \mathbb{N}$ which are not prime, then f is completely multiplicative.

B2. *If f is multiplicative and*

$$(g * \mu_\alpha)(p^k) - g(1)\mu_\alpha(p^k) - g(p^k) \neq 0 \tag{5}$$

for all integers $k \geq 2$ and prime p, then f is completely multiplicative.

C. *Assume that $g(1) = 1$ and*

$$(g * \mu_\alpha)(p^k) - g(p^k) + \alpha \neq 0 \tag{6}$$

for all primes p and all $k \in \mathbb{N}$, $k \geq 2$. If f is multiplicative and satisfies

$$f(g * \mu_\alpha) = fg * f^{-\alpha}, \tag{7}$$

then f is completely multiplicative.

Proposition 3. *Let $f, g \in \mathcal{A}_1$ and $\alpha \in \mathbb{R} \setminus \{0\}$. Assume that $g(1) = 1$ and*

$$(g * \mu_\alpha)(p^k) - g(p^k) + \alpha \neq 0 \tag{8}$$

for all primes p and all $k \in \mathbb{N}$, $k \geq 2$. If $f \in \mathcal{M}_1$ satisfies

$$f(g * \mu_\alpha) = fg * f^{-\alpha}, \tag{9}$$

then $f \in \mathcal{C}_1$.

Our first objective here is to generalize the above results in two directions: first by establishing results analogous to Propositions 1–3 for arithmetic functions of several variables, and second by establishing results analogous to Proposition 1 for discriminative products of more than two functions (of one variable).

In the final section, we give an affirmative answer the question whether the concept of discriminative products can be used to characterize multiplicative functions.

Part 1. Completely multiplicative functions of several variables

The following basic result is taken from [17, Proposition 2].

Proposition 4. *A function $F \in \mathcal{A}_r$ is completely multiplicative if and only if there exist completely multiplicative functions $f_1, \ldots, f_r \in \mathcal{C}_1$ (each of a single variable) such that*

$$F(n_1, \ldots, n_r) = f_1(n_1) \cdots f_r(n_r) \ (n_1, \ldots, n_r \in \mathbb{N}).$$

In this case $f_1(n) = F(n, 1, \ldots, 1)$, $\ldots$, $f_r(n) = F(1, \ldots, 1, n)$ for every $n \in \mathbb{N}$.

1. Characterizations based on discriminative products

We begin by deriving an r-variable extension of Proposition 1.

Theorem 1. *Let $F \in \mathcal{A}_r$.*

I. *Assume $F(1, \ldots, 1) \neq 0$. Then $F \in \mathcal{C}_r$ if and only if $F \in V_{\text{sep}}$ and F distributes over some discriminative product.*

II. *Assume $F \in \mathcal{M}_r$. Then $F \in \mathcal{C}_r$ if and only if $F \in V_{\text{sep}}$ and F distributes over some partially discriminative product.*

III. *Assume $F(1, \ldots, 1) = 1$. Then $F \in \mathcal{C}_r$ if and only if $F \in V_{\text{sep}}$ and F distributes over some semi-discriminative product.*

Proof. I. If $F \in \mathcal{C}_r$, by Proposition 4, we see that $F \in V_{\text{sep}}$, and F clearly distributes over any product.

Assume that $F \in V_{\text{sep}}$, and F distributes over a discriminative product $K = G * H$. There exist $A_1, \ldots, A_r \in \mathcal{A}_1$, $A_i(1) = A_j(1) \in \mathbb{R} \setminus \{0\}$ $(1 \le i, j \le r)$ such that

$$F(n_1, \ldots, n_r) = A_1(n_1) \cdots A_r(n_r).$$

By Proposition 4, it is enough to show that each $A_i \in \mathcal{C}_1$ for all $i = 1$, $\ldots$, r.

First, we show that $A_i(1) = 1$ $(i = 1, \ldots, r)$. The product $K = G * H$ being discriminative yields that $K(1, \ldots, 1) \ne 0$. Since F distributes over the product $K = G * H$, we have

$$\begin{aligned} F(1, \ldots, 1)K(1, \ldots, 1) &= F(1, \ldots, 1)G(1, \ldots, 1)F(1, \ldots, 1)H(1, \ldots, 1) \\ &= F(1, \ldots, 1)^2 K(1, \ldots, 1). \end{aligned}$$

From $F(1, \ldots, 1)K(1, \ldots, 1) \ne 0$ it follows that

$$1 = F(1, \ldots, 1) = A_1(1) \cdots A_r(1).$$

Using $A_i(1) = A_j(1)$ $(1 \le i, j \le r)$, we get

$$1 = A_1(1) = \cdots = A_r(1).$$

The distributive property yields

$$\begin{aligned} &A_1(n_1) \cdots A_r(n_r) \sum_{d_1 | n_1, \ldots, d_r | n_r} G(d_1, \ldots, d_r) H(n_1/d_1, \ldots, n_r/d_r) \\ &\quad = \sum_{d_1 | n_1, \ldots, d_r | n_r} A_1(d_1) \cdots A_r(d_r) G(d_1, \ldots, d_r) A_1(n_1/d_1) \times \cdots \\ &\qquad \times A_r(n_r/d_r) H(n_1/d_1, \ldots, n_r/d_r). \end{aligned}$$

Putting $n_i = n$, $n_k = 1$ $(k = 1, \ldots, r;\ k \ne i)$, we get

$$A_i(n) \sum_{d | n} G_i(d) H_i(n/d) = \sum_{d | n} A_i(d) G_i(d) A_i(n/d) H_i(n/d), \tag{10}$$

where G_i, H_i denote the i^{th} component functions of G, H, respectively. The relation (10) shows that A_i distributes over the product $G_i * H_i$ in $\mathcal{A}_1$. Since the discriminative property of the product $G * H$ implies that of $G_i * H_i$, invoking upon Langford's result (Proposition 1, part I), we deduce that $A_i \in \mathcal{C}_1$.

The proofs of parts II and III are similar but using Proposition 1, parts II and III, instead. □

The following corollary is immediate from the definition of firmly multiplicative functions which takes care of being variable separated.

Corollary 1. *Assume $F \in \mathcal{F}_r$. Then $F \in \mathcal{C}_r$ if and only if F distributes over some partially discriminative product.*

2. Characterizations involving Möbius functions

Recall four important concepts from the 1-dimensional case.

The log-derivation [13] is the operator $D : \mathcal{A}_1 \to \mathcal{A}_1$ defined by

$$(Df)(n) := f(n) \log n \quad (n \in \mathbb{N}, \quad f \in \mathcal{A}_1).$$

For $f \in \mathcal{A}_1$, $f(1) > 0$, the Rearick logarithmic operator of f (or logarithm of f; [11, 12]), denoted by $\operatorname{Log} f \in \mathcal{A}_1$, is defined via

$$\begin{aligned}(\operatorname{Log} f)(1) &= \log f(1),\\ (\operatorname{Log} f)(n) &= \frac{1}{\log n} \sum_{d|n} f(d) f^{-1}(n/d) \log d\\ &= \frac{1}{\log n} (Df * f^{-1})(n) \quad (n > 1),\end{aligned}$$

where D denotes the log-derivation.

For $h \in \mathcal{A}_1$, $h(1) \in \mathbb{R}$, the Rearick exponential $\operatorname{Exp} h$ is defined as the unique element $f \in \mathcal{A}_1$, $f(1) > 0$ such that $h = \operatorname{Log} f$.

For $f \in \mathcal{A}_1$, $f(1) > 0$ and $\alpha \in \mathbb{R}$, the α^{th} *power function* is defined as

$$f^\alpha = \operatorname{Exp}(\alpha \operatorname{Log} f).$$

Extending these four notions, for

$$F(n_1, \ldots, n_r) = A_1(n_1) \cdots A_r(n_r) \in V_{\text{sep}},$$

as given in (1), we define its

- *log-derivation*, $\Delta F \in V_{\text{sep}}$, by

$$\begin{aligned}(\Delta F)(n_1, \ldots, n_r) &:= (DA_1)(n_1) \cdots (DA_r)(n_r)\\ &= A_1(n_1) \log n_1 \cdots A_r(n_r) \log n_r;\end{aligned}$$

- *logarithm*, $L(F) \in V_{\text{sep}}$, provided $F(1, \ldots, 1) > 0$, by

$$\begin{aligned}L(F)(1, \ldots, 1) &= (\log A_1(1)) \cdots (\log A_r(1)),\\ L(F)(n_1, \ldots, n_r) &:= (\operatorname{Log} A_1)(n_1) \cdots (\operatorname{Log} A_r)(n_r)\\ &= \left(\frac{1}{\log n_1} \sum_{d_1|n_1} A_1(d_1) A_1^{-1}(n_1/d_1) \log d_1 \right) \times \cdots\\ &\quad \times \left(\frac{1}{\log n_r} \sum_{d_r|n_r} A_r(d_r) A_r^{-1}(n_r/d_r) \log d_r \right)\end{aligned}$$

$$= \left(\frac{1}{\log n_1}(DA_1 * A_1^{-1})(n_1)\right) \cdots \left(\frac{1}{\log n_r}(DA_r * A_r^{-1})(n_r)\right);$$

- *exponential*, $E(F) \in V_{\text{sep}}$, provided $F(1,\ldots,1) \in \mathbb{R}$, as the unique element $G \in V_{\text{sep}}$, $G(1,\ldots,1) > 0$ such that $F = L(G)$.
- α^{th} *power function* ($\alpha \in \mathbb{R}$), provided $F(1,\ldots,1) > 0$, by

$$F^\alpha = E\left(\alpha\, L(F)\right) \in V_{\text{sep}}.$$

For $\alpha \in \mathbb{R} \setminus \{0\}$ and $F \in V_{\text{sep}}$, as given in (1), observe that if F^α exists, then

$$F^\alpha = A_1^\alpha(n_1) \cdots A_r^\alpha(n_r) \in V_{\text{sep}}. \tag{11}$$

We now state and prove r-variable versions of Propositions 2 and 3.

Theorem 2. *Let $F, G \in \mathcal{A}_r$ with $G(1,\ldots,1) \neq 0$, and let $\alpha \in \mathbb{R} \setminus \{0\}$.*

A. *If $F \in \mathcal{C}_r$, then*

$$F\,(G * M_\alpha) = FG * FM_\alpha = FG * F^{-\alpha}.$$

B. *Assume that $F \in V_{\text{sep}}$ and*

$$F\,(G * M_\alpha) = FG * FM_\alpha. \tag{12}$$

B1. *If $F(1,\ldots,1) \neq 0$, and for $i \in \{1,\ldots,r\}$ the i^{th} component function satisfies*

$$(G_i * \mu_\alpha)(n) - G_i(1)\mu_\alpha(n) - G_i(n) \neq 0 \tag{13}$$

for all $n \in \mathbb{N}$ which are not prime, then $F \in \mathcal{C}_r$.

B2. *If $F \in \mathcal{M}_r$ and and for $i \in \{1,\ldots,r\}$ the i^{th} component function satisfies*

$$(G_i * \mu_\alpha)(p^k) - G_i(1)\mu_\alpha(p^k) - G_i(p^k) \neq 0 \tag{14}$$

for all integers $k \geq 2$ and prime p, then $F \in \mathcal{C}_r$.

C. *Assume that $F \in V_{\text{sep}}$, $G(1,\ldots,1) = 1$ and for $i \in \{1,\ldots,r\}$ the i^{th} component function satisfies*

$$(G_i * \mu_\alpha)(p^k) - G_i(p^k) + \alpha \neq 0 \tag{15}$$

for all primes p and all $k \in \mathbb{N}$, $k \geq 2$. If $F \in \mathcal{M}_r$ satisfies

$$F(G * M_\alpha) = FG * F^{-\alpha}, \tag{16}$$

then $F \in \mathcal{C}_r$.

Proof. A. The first equality is trivial, so we proceed to verify the second equality, which is equivalent to $FM_\alpha = F^{-\alpha}$. Since $F \in \mathcal{C}_r$, Proposition 4 tells us that $F(n_1,\ldots,n_r) = F_1(n_1)\cdots F_r(n_r)$, with each i^{th} component

function $F_i \in \mathcal{C}_1$. Invoking upon [3, Theorem 5], we have $F_i^{-\alpha} = \mu_\alpha F_i$. Combining with (11), we get

$$F^{-\alpha} = F_1^{-\alpha} \cdots F_r^{-\alpha} = (\mu_\alpha F_1) \cdots (\mu_\alpha F_r) = F M_\alpha.$$

B. The condition (14) indicates that F distributes over the product $G * M_\alpha$, while (13) and (14) are equivalent to the conditions of being discriminative and partially discriminative, respectively, the assertions B1 and B2 follow at once from Theorem 1, part I and II.

C. Since $F \in V_{\text{sep}}$, there are $A_1, \ldots, A_r \in \mathcal{A}_1$ such that

$$F(n_1, \ldots, n_r) = A_1(n_1) \cdots A_r(n_r).$$

Since $F \in \mathcal{M}_r$, each i^{th} component function $A_i \in \mathcal{M}_1$ and the relation (16) shows that

$$A_i(G_i * \mu_\alpha) = A_i G_i * A_i^{-\alpha} \quad (i = 1, \ldots, r).$$

By Proposition 3, we have that each $A_i \in \mathcal{C}_1$, and the result follows. □

Extensions of some well-known characterizations of completely multiplicative functions [8, Theorems 1.1 and 1.2] are simple applications of Theorem 2 as we show next.

Corollary 2. *Let $F \in \mathcal{M}_r \cap V_{\text{sep}}$ and $\alpha \in \mathbb{R} \setminus \{0\}$.*

I. *Then $F \in \mathcal{C}_r$ if and only if*

$$(M_\alpha F)^{-1} = M_{-\alpha} F.$$

II. *Assume that*

$$\mu_\alpha(p^k) + \alpha \neq 0$$

for all primes p and all $k \in \mathbb{N}$, $k \geq 2$. If

$$F M_\alpha = F^{-\alpha},$$

then $F \in \mathcal{C}_r$.

Proof. I. Observe first that

$$(M_\alpha F)^{-1} = M_{-\alpha} F \iff M_{-\alpha} F * M_\alpha F = I = FI = F\left(M_{-\alpha} * M_\alpha\right),$$

i.e., F distributes over the product $M_{-\alpha} * M_\alpha$. For prime p and integer $k \geq 2$, the i^{th} component of this product satisfies

$$(\mu_{-\alpha} * \mu_\alpha)(p^k) - \mu_{-\alpha}(1)\mu_\alpha(p^k) - \mu_{-\alpha}(p^k) = -\mu_\alpha(p^k) - \mu_{-\alpha}(p^k) \neq 0.$$

The assertion follows at once from Theorem 2, parts B2 and A.

The assertion II is immediate from taking $G = I$ in part C of Theorem 2. □

Part 2. Generalized discriminative products of functions of one variable

We begin with some preliminaries.

3. Definition and results

Let $s \in \mathbb{N}$, $s \geq 2$ and let $g_1, g_2, \ldots, g_s \in \mathcal{A}_1 \setminus \{0\}$. We say that the product $k := g_1 * g_2 * \cdots * g_s$ is

- *s-fold discriminative*, or "s.d." for short, if the relation
$$k(n) = \sum_{j=1}^{s} g_1(1) \cdots g_{j-1}(1) g_j(n) g_{j+1}(1) \cdots g_s(1) \tag{17}$$
holds only when n is prime;
- *s-fold partially discriminative*, or "s.p.d.", if for every prime power p^i $(i \in \mathbb{N})$, the relation
$$k(p^i) = \sum_{j=1}^{s} g_1(1) \cdots g_{j-1}(1) g_j(p^i) g_{j+1}(1) \cdots g_s(1) \tag{18}$$
implies that $i = 1$;
- *s-fold semi-discriminative*, or "s.s.d.", if the relation
$$k(n) = \sum_{j=1}^{s} g_1(1) \cdots g_{j-1}(1) g_j(n) g_{j+1}(1) \cdots g_s(1) \tag{19}$$
holds only when $n = 1$ or n is prime.

Our main theorem in this part reads:

Theorem 3. *Let $s \in \mathbb{N}$, $s \geq 2$, let $g_1, g_2, \cdots, g_s \in \mathcal{A}_1 \setminus \{0\}$, and let $k := g_1 * g_2 * \cdots * g_s$.*

(i) *If f is completely multiplicative, then f distributes over k, i.e.,*
$$f(g_1 * g_2 * \cdots * g_s) = fg_1 * fg_2 * \cdots * fg_s. \tag{20}$$

(ii) *If $f(1) \neq 0$ and f distributes over an s.d. product, then $f(1)$ is an $(s-1)^{th}$ root of unity and f is completely multiplicative.*

(iii) *If f is multiplicative and distributes over an s.p.d. product, then f is completely multiplicative.*

(iv) *If $f(1) = 1$ and f distributes over an s.s.d. product, then f is completely multiplicative.*

Proof. Part (i) is trivial. We proceed to prove Part (ii). Assume that f distributes over an s.d. product $k = g_1 * g_2 * \cdots * g_s$. We claim that $k(1) \neq 0$. For if
$$0 = k(1) = g_1(1) g_2(1) \cdots g_s(1),$$

then $g_i(1) = 0$ for some i, so that (17) holds at $n = 1$, contradicting the fact that k is s.d. Evaluating (20) at $n = 1$, we get

$$f(1)k(1) = (f(1))^r\, k(1).$$

Since $f(1)k(1) \neq 0$, we have $f(1)^{r-1} = 1$.

To show that f is completely multiplicative, it suffices to show that if $p_1, p_2, \ldots, p_t$ is any choice of primes (distinct or not), then

$$f(p_1 \cdots p_t) = f(p_1) \cdots f(p_t).$$

We proceed by induction on $t \in \mathbb{N}$. This is trivial for $t = 1$, so assume that $t \geq 2$. Since f distributes over an s.d. product $k = g_1 * g_2 * \cdots * g_s$, we have

$$f(p_1 \cdots p_t) \sum_{d_1 \cdots d_s = p_1 \cdots p_t} g_1(d_1) \cdots g_s(d_s) = \sum_{d_1 \cdots d_s = p_1 \cdots p_t} (fg_1)(d_1) \cdots (fg_s)(d_s).$$

Using the induction hypothesis, we get

$$\begin{aligned} f(p_1 \cdots p_t) &\Bigg(\sum_{\substack{d_1 \cdots d_s = p_1 \cdots p_t \\ d_i \neq p_1 \cdots p_t \text{ for all } i \in \{1,\ldots,s\}}} g_1(d_1) \cdots g_s(d_s) + \sum_{\substack{d_1 \cdots d_s = p_1 \cdots p_t \\ d_i = p_1 \cdots p_t \text{ for some } i \in \{1,\ldots,s\}}} g_1(d_1) \cdots g_s(d_s) \Bigg) \\ &= f(p_1) \cdots f(p_t) \sum_{\substack{d_1 \cdots d_s = p_1 \cdots p_t \\ d_i \neq p_1 \cdots p_t \text{ for all } i \in \{1,\ldots,s\}}} g_1(d_1) \cdots g_s(d_s) \\ &\quad + f(p_1 \cdots p_t) f(1)^{s-1} \sum_{\substack{d_1 \cdots d_s = p_1 \cdots p_t \\ d_i = p_1 \cdots p_t \text{ for some } i \in \{1,\ldots,s\}}} g_1(d_1) \cdots g_s(d_s). \end{aligned}$$

Using the fact that $f(1)^{s-1} = 1$, we get

$$\big(f(p_1 \cdots p_t) - f(p_1) \cdots f(p_t)\big) \sum_{\substack{d_1 \cdots d_s = p_1 \cdots p_t \\ d_i \neq p_1 \cdots p_t \text{ for all } i \in \{1,\ldots,s\}}} g_1(d_1) \cdots g_s(d_s) = 0.$$

Since

$$\begin{aligned} &\sum_{\substack{d_1 \cdots d_s = p_1 \cdots p_t \\ d_i \neq p_1 \cdots p_t \text{ for all } i \in \{1,\ldots,s\}}} g_1(d_1) \cdots g_s(d_s) \\ &\qquad = k(p_1 \cdots p_t) - \sum_{\substack{d_1 \cdots d_s = p_1 \cdots p_t \\ d_i = p_1 \cdots p_t \text{ for some } i \in \{1,\ldots,s\}}} g_1(d_1) \cdots g_s(d_s) \end{aligned}$$

and k is s.d., we must have

$$\sum_{\substack{d_1 \cdots d_s = p_1 \cdots p_t \\ d_i \neq p_1 \cdots p_t \text{ for all } i \in \{1,\ldots,s\}}} g_1(d_1) \cdots g_s(d_s) \neq 0,$$

which yields at once that $f(p_1 \cdots p_t) = f(p_1) \cdots f(p_t)$, as desired.

(iii) Assume that f distributes over an s.p.d. product $k = g_1 * g_2 * \cdots * g_s$. Since f is multiplicative, to show that it is completely multiplicative, it suffices to show that

$$f(p^m) = f(p)^m$$

for all primes p and for all $m \in \mathbb{N}$. The remaining proof is by induction on m similar to that of the last half of Part (ii) but making use of k being s.p.d. instead.

Part (iv) follows through the same proof as in the last half of Part (ii). □

Applying the results of Theorem 3 to the Souriau–Hsu–Möbius (SHM) functions, we have the following generalization of Proposition 2.

Corollary 3. *Let $s \in \mathbb{N}$, $s \geq 2$, let $f, g \in \mathcal{A}_1$ with $g(1) \neq 0$, and let $\alpha_1, \ldots, \alpha_s \in \mathbb{R} \setminus \{0\}$.*

A. *If f is completely multiplicative, then*

$$f\left(g * \mu_{\alpha_1} * \cdots * \mu_{\alpha_s}\right) = fg * f\mu_{\alpha_1} * \cdots * f\mu_{\alpha_s} = fg * f^{-\alpha_1} * \cdots * f^{\alpha_s}.$$

B. *Assume that*

$$f\left(g * \mu_{\alpha_1} * \cdots * \mu_{\alpha_s}\right) = fg * f\mu_{\alpha_1} * \cdots * f\mu_{\alpha_s}. \tag{21}$$

B1. *If $f(1) > 0$ and*

$$(g * \mu_{\alpha_1} * \cdots * \mu_{\alpha_s})(n) - g(n) - g(1) \sum_{j=1}^{s} \mu_{\alpha_j}(n) \neq 0 \tag{22}$$

for all $n \in \mathbb{N}$ which are not prime, then f is completely multiplicative.

B2. *If f is multiplicative and*

$$(g * \mu_{\alpha_1} * \cdots * \mu_{\alpha_s})(p^k) - g(p^k) - g(1) \sum_{j=1}^{s} \mu_{\alpha_j}(p^k) \neq 0 \tag{23}$$

for all integers $k \geq 2$ and prime p, then f is completely multiplicative.

C. *Assume that $g(1) = 1$ and*

$$(g * \mu_{\alpha_1} * \cdots * \mu_{\alpha_s})(p^k) - g(p^k) + \alpha_1 + \cdots + \alpha_s \neq 0 \tag{24}$$

for all primes p and all $k \in \mathbb{N}$, $k \geq 2$. If $f \in \mathcal{M}_1$ satisfies

$$f(g * \mu_{\alpha_1} * \cdots * \mu_{\alpha_s}) = fg * f^{-\alpha_1} * \cdots * f^{-\alpha_s}, \tag{25}$$

then f is completely multiplicative.

Proof. If $f \in \mathcal{C}_1$, it is well known [3, 8] that $f^{-\alpha} = f\mu_\alpha$. Using this relation, the assertion A is immediate.

B. Since the two conditions (22) and (23) are equivalent to the conditions of being s.d. and s.p.d., respectively, the results in B1 and B2 follow readily from Theorem 3, parts (ii) and (iii), respectively.

C. Since f is multiplicative, we need to show that $f(p^k) = f(p)^k$ for all primes p and for all $k \in \mathbb{N}$. The case $k = 1$ being trivial, we proceed by induction to show that it holds for all $k \geq 2$. If $f(p^i) = f(p)^i$ for $0 \leq i \leq k-1$, from the proof of [10, Theorem 1, p. 317], we have

$$f^\alpha(p^i) = f(p)^i \mu_{-\alpha}(p^i) = \binom{-\alpha}{i}(-1)^i f(p)^i = f(p)^i \binom{\alpha+i-1}{i} \tag{26}$$

and

$$f^\alpha(p^k) = f(p)^k \left(\mu_{-\alpha}(p^k) - \alpha\right) + \alpha f(p^k). \tag{27}$$

Rewriting (25) in an equivalent form as

$$f(g * \mu_{\alpha_1} * \cdots * \mu_{\alpha_s}) * f^{\alpha_1} * \cdots * f^{\alpha_s} = fg$$

and evaluating at p^k, we get

$$\begin{aligned} fg(p^k) &= f(g * \mu_{\alpha_1} * \cdots * \mu_{\alpha_s})(p^k) \\ &\quad + f(g * \mu_{\alpha_1} * \cdots * \mu_{\alpha_s})(p^{k-1})(f^{\alpha_1} * \cdots * f^{\alpha_s})(p) + \cdots \\ &\quad + f(g * \mu_{\alpha_1} * \cdots * \mu_{\alpha_s})(p)(f^{\alpha_1} * \cdots * f^{\alpha_s})(p^{k-1}) \\ &\quad + (f^{\alpha_1} * \cdots * f^{\alpha_s})(p^k). \end{aligned}$$

Using the induction hypothesis together with (26) and (27), we have

$$\begin{aligned} fg(p^k) &= f(p^k)(g * \mu_{\alpha_1} * \cdots * \mu_{\alpha_s})(p^k) \\ &\quad + f(p)^k \Big\{ (g * \mu_{\alpha_1} * \cdots * \mu_{\alpha_s})(p^{k-1})(\mu_{-\alpha_1} * \cdots * \mu_{-\alpha_s})(p) + \cdots \\ &\qquad + (g * \mu_{\alpha_1} * \cdots * \mu_{\alpha_s})(p)(\mu_{-\alpha_1} * \cdots * \mu_{-\alpha_s})(p^{k-1}) \\ &\qquad + (\mu_{-\alpha_1} * \cdots * \mu_{-\alpha_s})(p^k) \Big\} \\ &\quad - (\alpha_1 + \cdots + \alpha_s) f(p)^k \\ &\quad + (\alpha_1 + \cdots + \alpha_s) f(p^k) \\ &= f(p^k) \Big\{ (g * \mu_{\alpha_1} * \cdots * \mu_{\alpha_s})(p^k) + (\alpha_1 + \cdots + \alpha_s) \Big\} \\ &\quad + f(p)^k \Big\{ (g * \mu_{\alpha_1} * \cdots * \mu_{\alpha_s} * \mu_{-\alpha_1} * \cdots * \mu_{-\alpha_s})(p^k) \\ &\qquad - (g * \mu_{\alpha_1} * \cdots * \mu_{\alpha_s})(p^k) \\ &\qquad - (\alpha_1 + \cdots + \alpha_s) \Big\}. \end{aligned}$$

Thus,

$$f(p^k)\Big\{(g*\mu_{\alpha_1}*\cdots*\mu_{\alpha_s})(p^k)+(\alpha_1+\cdots+\alpha_s)-g(p^k)\Big\}$$
$$=f(p)^k\Big\{(g*\mu_{\alpha_1}*\cdots*\mu_{\alpha_s})(p^k)+(\alpha_1+\cdots+\alpha_s)-g(p^k)\Big\},$$

and so (24) assures us that $f(p^k)=f(p)^k$. □

To end this section, we mention a nice application of the concept of s.s.d. This is an extension of [10, Corollary 1], which gives a characterization of completely multiplicative function via generalized von Mangoldt function. Recall that the classical von Mangoldt function of order $k\in\mathbb{N}$, [5], is defined by

$$\Lambda_k(n)=(\mu*\log^k)(n)=\sum_{d|n}\mu(d)\log^k(n/d).$$

For $s\in\mathbb{N}$ and $\alpha_1,\ldots,\alpha_s\in\mathbb{R}$, the s-fold SHM–von Mangoldt function of order $k\in\mathbb{N}$ is defined as

$$\Lambda_{k,\alpha}(n):=\left(\mu_{\alpha_1}*\cdots*\mu_{\alpha_s}*\log^k\right)(n)=\sum_{d_1\cdots d_s d=n}\mu_{\alpha_1}(d_1)\cdots\mu_{\alpha_s}(d_s)\log^k(d).$$

Proposition 5. *Let $k,s\in\mathbb{N}$, let $\alpha_1,\ldots,\alpha_s\in(0,1]$. Then an arithmetic function f with $f(1)=1$ is completely multiplicative if and only if f distributes over $\Lambda_{k,\alpha}$, i.e., if and only if*

$$f\Lambda_{k,\alpha}=\left(f\mu_{-\alpha_1}\right)^{-1}*\cdots*\left(f\mu_{-\alpha_s}\right)^{-1}*f\log^k.$$

Proof. The proof is much the same as that of [10, Corollary 1] using the following facts:

(i) $\Lambda_{k,\alpha}=\Lambda_{k-1}\log*\mu_{\alpha_1-1}*\cdots*\mu_{\alpha_s-1}+\Lambda_{k-1}*\Lambda_{1,\alpha}$;
(ii) $\Lambda_{k,\alpha}(n)\geq 0$ for all $n\in\mathbb{N}$;
(iii) the product $\Lambda_{k,\alpha}*\mu_{-\alpha_1}*\cdots*\mu_{-\alpha_s}$ is s.s.d. □

4. Unitary products

As seen from our earlier investigation, the concept of discriminative products can be used to characterize completely multiplicative functions. It is natural to ask whether similar things can be done for multiplicative functions. We give an affirmative answer in this last section. The unitary product of two arithmetic functions $f,g\in\mathcal{A}_1$ is defined by

$$(f\circ g)(n)=\sum_{\substack{ij=n\\(i,j)=1}}f(i)g(j).$$

In [16], Subbarao gave the following characterization of multiplicative functions using the unitary product.

Proposition 6. *The arithmetic function f satisfies*

$$f(g \circ h) = fg \circ fh$$

for all $g, h \in \mathcal{A}_1$ if and only if f is multiplicative.

In order to extend this result of Subbarao along the direction of the last section, we introduce the concept of u-discriminative product. For $s \in \mathbb{N}$, $s \geq 2$, we say that the product

$$k := g_1 \circ g_2 \circ \cdots \circ g_s$$

is *s-fold u-discriminative*, or s.u.d., if the relation

$$k(n) = \sum_{j=1}^{s} g_1(1) \cdots g_{j-1}(1) g_j(n) g_{j+1}(1) \cdots g_s(1) \tag{28}$$

holds only when n is a prime power p^i $(i \in \mathbb{N})$.

Our final result is:

Theorem 4. *Let $s \in \mathbb{N}$, $s \geq 2$ and $f, g_1, g_2, \cdots, g_s \in \mathcal{A}_1 \setminus \{0\}$.*

(i) *If f is multiplicative, then f distributes over $k := g_1 \circ g_2 \circ \cdots \circ g_s$.*

(ii) *If $f(1) \neq 0$ and f distributes over an s.u.d. product k, then $f(1)$ is a root of $z^{s-1} - 1 = 0$ and f is multiplicative.*

Proof. Assertion (i) follows readily from f being multiplicative and the product being unitary.

(ii) If f distributes over an s.u.d. product $k = g_1 \circ g_2 \circ \cdots \circ g_s$, then we claim that $k(1) \neq 0$. For otherwise

$$0 = k(1) = g_1(1) g_2(1) \cdots g_s(1)$$

implies that $g_i(1) = 0$ for some i, i.e, (28) holds with $n = 1$, contradicting the fact that k is s.u.d. Evaluating fk at $n = 1$, we get

$$f(1)k(1) = \{f(1)\}^r g_1(1) g_2(1) \cdots g_s(1) = \{f(1)\}^s k(1),$$

and using $f(1)k(1) \neq 0$, we have $f(1)^{s-1} = 1$.

To prove that f is multiplicative, it suffices to show that if p_1, p_2, …, p_t is any choice of distinct primes and $a_i \in \mathbb{N}$ $(i = 1, \ldots, t)$, then

$$f(p_1^{a_1} \cdots p_t^{a_t}) = f(p_1^{a_1}) \cdots f(p_t^{a_t}).$$

The remaining proof is by induction on t similar to that of the last half of the proof of Theorem 3, part (ii), but making use of k being s.u.d. with the product being unitary instead. □

References

1. E. Alkan, A. Zaharescu and M. Zaki, *Arithmetical functions in several variables*, International Journal of Number Theory **1** (2005), no. 3, 383–399.
2. T. C. Brown, L. C. Hsu, J. Wang and P. J.-S. Shiue, *On a certain kind of generalized number-theoretical Möbius function*, Math. Sci. **25** (2000), no. 2, 72–77.
3. P. Haukkanen, *On the real powers of completely multiplicative arithmetical functions*, Nieuw Arch. Wiskd. **15** (1997), no. 1-2, 73–77.
4. P. Haukkanen, *On characterizations of completely multiplicative arithmetical functions*, in: Number Theory (Turku, 1999), de Gruyter, Berlin, 2001, pp. 115–123.
5. A. Ivić, *An application of Dirichlet series to certain arithmetical functions*, Math. Balkanica **3** (1973), 158–165.
6. J. Lambek, *Arithmetic functions and distributivity*, Amer. Math. Monthly **73** (1966), 969–973.
7. E. Langford, *Distributivity over the Dirichlet product and completely multiplicative arithmetical functions*, Amer. Math. Monthly **80** (1973), 411–414.
8. V. Laohakosol, N. Pabhapote and N. Wechwiriyakul, *Characterizing completely multiplicative functions by generalized Möbius functions*, Int. J. Math. Math. Sci. **29** (2002), no. 11, 633–639.
9. P. J. McCarthy, *Introduction to Arithmetical Functions*, Springer-Verlag, New York, 1985.
10. N. Pabhapote and V. Laohakosol, *Distributive property of completely multiplicative functions*, Lithuanian Math. J. **50** (2010), no. 3, 312–322.
11. D. Rearick, *Operators on algebras of arithmetic functions*, Duke Math. J. **35** (1968), 761–766.
12. D. Rearick, *The trigonometry of numbers*, Duke Math. J. **35**, 767–776.
13. H. N. Shapiro, *Introduction to the Theory of Numbers*, John Wiley and Sons, New York, 1983.
14. R. Sivaramakrishnan, *Classical Theory of Arithmetic Functions*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 126, Marcel Dekker, New York, 1989.
15. Jean-Marie Souriau, *Généralisation de certaines formules arithmétiques d' inversion. Applications*, Revue Scientific (Rev. Rose Illus.) **82** (1944), 204–211.
16. M.V. Subbarao, *Arithmetic functions and distributivity*, Amer. Math. Monthly **75** (1968), 984–988.
17. L. Tóth, *Multiplicative Arithmetic Functions of Several Variables: a Survey*, in vol. Mathematics without Boundaries, Surveys in Pure Mathematics, Th. M. Rassias, P. Pardalos (Editors), Springer, New York, 2014, pp. 483–514.

Department of Mathematics, Faculty of Science, Kasetsart University, Bangkok, 10900, Thailand
Email: fscivil@ku.ac.th

Department of Mathematics, Faculty of Science, University of the Thai Chamber of Commerce, Bangkok, 10400, Thailand
Email: nittiya_pab@utcc.ac.th

Voronoï's Impact on Modern Science, Book 6, Volume 1
Proceedings of the Sixth International Conference on Analytic Number Theory and Spatial Tessellations

EXTENSIONS OF SCHWARZ'S THEOREM

Vichian Laohakosol and Angkana Sripayap (*Thailand*)

Abstract. A result of Schwarz about transcendence of certain Lambert series evaluated at rational points is extended in two directions. First, the exponential power is replaced by a more general integer-valued function. Second, an algebraic independence criterion of such functions is derived.

Introduction

In [7], Schwarz proved the irrationality of a number written as a Lambert series of the form

$$G_k\left(\frac{b}{t}\right)=\sum_{n\geq 0}\frac{b^{k^n}}{t^{k^n}-b^{k^n}},$$

where k, t, b are integers satisfying the inequalities

$$k\geq 2,\quad t\geq 2,\quad \text{and}\quad 0<b<t^{1-1/k},$$

and proved the transcendence of the same number under some stronger inequalities for the parameters

$$k>2,\quad t\geq 2,\quad \text{and}\quad 0<b<t^{1-5/(2k)}.$$

These results extend the ones obtained earlier by Chowla [3], Erdös [4] and Golomb [5].

In the present work, we generalize the above results of Schwarz even further by proving the following theorem.

Theorem 1. *Let b, $t\geq 2$ be positive integers, let $f:\mathbb{N}\to\mathbb{N}$ be a strictly increasing function such that $f(n)\mid f(n+1)$, let $g(n)=f(n+1)/f(n)$, and let*

$$G_f\left(\frac{b}{t}\right):=\sum_{n\geq 0}\frac{b^{f(n)}}{t^{f(n)}-b^{f(n)}}. \tag{1}$$

Received July 23, 2018; revised November 6, 2018.
2010 *Mathematics Subject Classification.* 11J91, 33E20.
Key words and phrases. Lambert series, transcendence, Liouville number, algebraic independence.

(i) *If* $\lim_{n\to\infty} g(n) = \ell > 2$, *and* $0 < b < t^{1-A}$ *for some* $2/\ell < A < 1$, *then the number* $G_f(b/t)$ *is transcendental.*

(ii) *If* $\lim_{n\to\infty} g(n) = \infty$, *and* $0 < b < t^{1-A}$ *for some* $0 < A < 1$, *then* $G_f(b/t)$ *is a transcendental number which is a Liouville number.*

We refer to the numbers under the form (1) as **Lambert numbers**.

As with Schwarz in [7], the proof of Theorem 1 makes use of Roth's theorem which we now quote.

Theorem 2 ([6]). *If* α *is an algebraic number of degree* $n \geq 2$. *Then for each* $\varepsilon > 0$, *the inequality*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\varepsilon}}$$

has only finitely many solutions in rationals p/q.

In order to be self-contained, we recall the definition of Liouville numbers.

Definition 1 ([2, Definition 1.1, p. 4]). *Let* x *be a real number. If for any positive real number* ω *there exist a rational number* p/q *with* $p, q \in \mathbb{N}$, $q > 1$, *such that*

$$0 < \left|x - \frac{p}{q}\right| < \frac{1}{q^{\omega}},$$

then x *is called a Liouville number.*

The proof of algebraic independence makes use of the following result of Adams.

Theorem 3 ([1]). *Let* $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R} \setminus \{0\}$. *Assume that there are infinitely many integers* $P_{N,j}$, $Q_{N,j}$ $(\in \mathbb{N})$ $(\neq 0)$ $(1 \leq j \leq n,\ N \in \mathbb{N})$ *such that*

(i) $Q_{N,j} \to \infty$ $(N \to \infty,\ 1 \leq j \leq n)$;

(ii)
$$\lim_{N\to\infty} \frac{\left|\alpha_{j-1} - \frac{P_{N,j-1}}{Q_{N,j-1}}\right|}{\left|\alpha_j - \frac{P_{N,j}}{Q_{N,j}}\right|} = 0 \quad (2 \leq j \leq n), \tag{2}$$
and

(iii) *for each positive number* E, *there exists* $N_0 = N_0(E) \in \mathbb{N}$ *such that*
$$0 < \left|\alpha_j - \frac{P_{N,j}}{Q_{N,j}}\right| \leq \frac{1}{(Q_{N,1}Q_{N,2}\cdots Q_{N,j})^E} \quad (N \geq N_0,\ 1 \leq j \leq n). \tag{3}$$

Then α_1, α_2, $\ldots$, α_n *are* $\mathbb{Q}$*-algebraically independent.*

Our algebraic independence results for transcendental numbers, appearing in Theorem 1 (ii), is

Theorem 4. *Let b, $t \geq 2$ be positive integers. For $i \in \{1, \ldots, k\}$, $k \geq 2$, let $f_i : \mathbb{N} \to \mathbb{N}$ be strictly increasing functions such that $f_i(n) \mid f_i(n+1)$, let*

$$g_i(n) := \frac{f_i(n+1)}{f_1(n) + \cdots + f_k(n)},$$

and let

$$\alpha_i := G_{f_i}\left(\frac{b}{t}\right) = \sum_{n=0}^{\infty} \frac{b^{f_i(n)}}{t^{f_i(n)} - b^{f_i(n)}}.$$

For $i \in \{1, 2, \ldots, k\}$, assume that $\lim_{n\to\infty} g_i(n) = \infty$ and that $0 < b < t^{1-A}$ for some $0 < A < 1$. If

$$h_{n,i} := f_{i-1}(n) - f_i(n) \to \infty \quad (n \to \infty;\ i \in \{2, 3, \ldots, k\}),$$

then $\alpha_1, \ldots, \alpha_k$ are algebraically independent over $\mathbb{Q}$.

1. Proof of Theorem 1

For $N \in \mathbb{N}$, $N > 1$, denote the truncated Nth partial sum by

$$\sum_{n \leq N} \frac{b^{f(n)}}{t^{f(n)} - b^{f(n)}} =: \frac{a_N}{q_N},$$

where $q_N = t^{f(N)} - b^{f(N)} < t^{f(N)}$. Since $0 < b/t < t^{-A} < 1$ and f is strictly increasing, for sufficiently large N, we have

$$\begin{aligned}\left|G_f\left(\frac{b}{t}\right) - \frac{a_N}{q_N}\right| &= \sum_{n>N} \frac{(b/t)^{f(n)}}{1 - (b/t)^{f(n)}} \\ &< 2\sum_{n>N}\left(\frac{b}{t}\right)^{f(n)} < 2\sum_{m \geq f(N+1)}\left(\frac{b}{t}\right)^{m} \\ &< 2\left(\frac{b}{t}\right)^{f(N+1)} \sum_{m \geq 0}\left(\frac{b}{t}\right)^{m} = 2\beta\left(\frac{b}{t}\right)^{f(N+1)}, \qquad (4)\end{aligned}$$

where $\beta = \frac{1}{1-(b/t)}$. From $(b/t)^{f(N+1)} < t^{-Af(N+1)} < q_N^{-Ag(N)}$, we get

$$\left|G_f\left(\frac{b}{t}\right) - \frac{a_N}{q_N}\right| < \frac{2\beta}{q_N^{Ag(N)}}. \qquad (5)$$

(i) Since $\lim_{n\to\infty} g(n) = \ell > 2$ and $2/\ell < A < 1$, let $\varepsilon := \frac{1}{2}(A\ell - 2) > 0$. Thus, for all N sufficiently large, we get $q_N^{Ag(N)} > 2\beta\, q_N^{2+\varepsilon}$, and (5) gives

$$\left| G_f\left(\frac{b}{t}\right) - \frac{a_N}{q_N} \right| < \frac{1}{q_N^{2+\epsilon}}.$$

By Roth's theorem, we deduce that $G_f\,(b/t)$ is either a rational or a transcendental number.

To rule out the former possibility, we argue as in [7]. Assume, on the contrary, that

$$G_f\left(\frac{b}{t}\right) := \sum_{n\geq 0} \frac{b^{f(n)}}{t^{f(n)} - b^{f(n)}} = \frac{a}{q}$$

is rational. For $n \in \mathbb{N}$, let

$$D_1 := q\left(t^{f(N)} - b^{f(N)}\right) G_f\left(\frac{b}{t}\right) = a\left(t^{f(N)} - b^{f(N)}\right) \in \mathbb{N},$$

$$D_2 := q\left(t^{f(N)} - b^{f(N)}\right) \sum_{n\leq N} \frac{b^{f(n)}}{t^{f(n)} - b^{f(n)}} \in \mathbb{N}$$

because $f(n) \mid f(n+1)$ implies that $\left(t^{f(n)} - b^{f(n)}\right) \mid \left(t^{f(N)} - b^{f(N)}\right)$ for $n \in \{1, 2, \ldots, N\}$. Thus,

$$D := D_1 - D_2 = q\left(t^{f(N)} - b^{f(N)}\right) \sum_{n>N} \frac{b^{f(n)}}{t^{f(n)} - b^{f(n)}} \in \mathbb{N}. \tag{6}$$

On the other hand, since $b/t < t^{-A}$, for N sufficiently large, we have

$$\begin{aligned} \sum_{n>N} \frac{b^{f(n)}}{t^{f(n)} - b^{f(n)}} &= \sum_{n>N} \frac{(b/t)^{f(n)}}{1 - (b/t)^{f(n)}} \\ &< 2\sum_{n>N} \left(\frac{b}{t}\right)^{f(n)} < 4\left(\frac{b}{t}\right)^{f(N+1)} < \frac{4}{t^{Af(N+1)}} \end{aligned}$$

yielding

$$0 < D < \frac{4q}{t^{Af(N+1)-f(N)}} = \frac{4q}{t^{f(N)(Af(N+1)/f(N)-1)}} \to 0 \quad (N \to \infty),$$

because

$$\frac{Af(N+1)}{f(N)} - 1 \to A\ell - 1 > 1,$$

which contradicts (6).

(ii) Let $\varepsilon \in (0,1)$ be arbitrary. Since $g(n) \to \infty$ $(n \to \infty)$, there exists $N_0 \in \mathbb{N}$ such that

$$q^{Ag(N)} > 2\beta\, q^{2+\varepsilon} \quad (N \geq N_0).$$

Using (5) and Roth's theorem (Theorem 2), we deduce at once that $G_f(b/t)$ is either rational or transcendental. The possibility of being rational is ruled out by the same proof as in the previous case (i). To show that $G_f(b/t)$ is a Liouville number, let $\omega > 0$. Since $\lim_{n\to\infty} g(n) = \infty$, there exists $N_\omega \in \mathbb{N}$ such that

$$q^{Ag(N)} > 2\beta\, q^{\omega} \quad (N \geq N_\omega). \tag{7}$$

Using (5) and Roth's theorem (Theorem 2), from Definition 1, we deduce at once that $G_f(b/t)$ is a Liouville number.

2. Proof of Theorem 4

To make use of Theorem 3, we proceed by checking the stated conditions successively. For fixed $N \in \mathbb{N}$, let

$$\frac{P_{N,i}}{Q_{N,i}} := \sum_{n=0}^{N} \frac{b^{f_i(n)}}{t^{f_i(n)} - b^{f_i(n)}},$$

where $Q_{N,i} = t^{f_i(N)} - b^{f_i(N)}$. From

$$\begin{aligned} P_{N,i} &= b^{f_i(1)} \left(\frac{t^{f_i(N)} - b^{f_i(N)}}{t^{f_i(1)} - b^{f_i(1)}} \right) + \cdots \\ &\quad + b^{f_i(N-1)} \left(\frac{t^{f_i(N)} - b^{f_i(N)}}{t^{f_i(N-1)} - b^{f_i(N-1)}} \right) + b^{f_i(N)}, \\ Q_{N,i}\,\alpha_i &= \left(t^{f_i(N)} - b^{f_i(N)}\right) \sum_{n=0}^{\infty} \frac{b^{f_i(n)}}{t^{f_i(n)} - b^{f_i(n)}} \\ &= P_{N,i} + \left(t^{f_i(N)} - b^{f_i(N)}\right) \sum_{n=N+1}^{\infty} \frac{b^{f_i(n)}}{t^{f_i(n)} - b^{f_i(n)}}, \end{aligned}$$

we see that $P_{N,i} - Q_{N,i}\,\alpha_i \neq 0$.

Since $0 < b/t < t^{-A} < 1$, each f_i is strictly increasing with

$$\frac{f_i(n+1)}{f_i(n)} \to \infty \quad (n \to \infty),$$

and the algebraic independence of Lambert series is not affected by omitting some initial terms of the series, we assume without loss of generality

that each $f_i(0)$ is so taken that the value of its Lambert series is < 1, i.e.,

$$\alpha_i = G_{f_i}(b/t) = \sum_{n=0}^{\infty} \frac{b^{f_i(n)}}{t^{f_i(n)} - b^{f_i(n)}} < 1.$$

Thus, for each $i \in \{1, \ldots, k\}$, we have

$$M_{N,i} := \max\{P_{N,i}, Q_{N,i}\} = Q_{N,i} \to \infty \quad (N \to \infty),$$

i.e., condition (i) of Theorem 3 holds.

Consider, for $2 \le j \le k$,

$$\begin{aligned}
\frac{\left|\alpha_{i-1} - \frac{P_{N,i-1}}{Q_{N,i-1}}\right|}{\left|\alpha_i - \frac{P_{N,i}}{Q_{N,j}}\right|} &= \frac{\sum\limits_{n=N+1}^{\infty} \frac{(b/t)^{f_{i-1}(n)}}{1-(b/t)^{f_{i-1}(n)}}}{\sum\limits_{n=N+1}^{\infty} \frac{(b/t)^{f_i(n)}}{1-(b/t)^{f_i(n)}}} \\
&= \frac{\frac{(b/t)^{f_{i-1}(N+1)}}{1-(b/t)^{f_{i-1}(N+1)}} \left(1 + (b/t)^{f_{i-1}(N+2)-f_{i-1}(N+1)} \left(\frac{1-(b/t)^{f_{i-1}(N+1)}}{1-(b/t)^{f_{i-1}(N+2)}}\right) + \cdots\right)}{\frac{(b/t)^{f_i(N+1)}}{1-(b/t)^{f_i(N+1)}} \left(1 + (b/t)^{f_i(N+2)-f_i(N+1)} \left(\frac{1-(b/t)^{f_i(N+1)}}{1-(b/t)^{f_i(N+2)}}\right) + \cdots\right)} \\
&\le \left(\frac{b}{t}\right)^{h_{N+1,i}} \cdot \frac{1-(b/t)^{f_i(N+1)}}{1-(b/t)^{f_{i-1}(N+1)}} \cdot B \to 0 \quad (N \to \infty),
\end{aligned}$$

where

$$B = 1 + (b/t)^{f_{i-1}(N+2)-f_{i-1}(N+1)} \left(\frac{1-(b/t)^{f_{i-1}(N+1)}}{1-(b/t)^{f_{i-1}(N+2)}}\right) + \cdots < \infty,$$

because $h_{N+1,i} \to \infty$ and $0 < b/t < t^{-A} < 1$. Thus, condition (ii) of Theorem 3 holds.

By the same arguments as in the beginning of the proof of Theorem 1, for N sufficiently large and for $i \in \{1, \ldots, k\}$, from (4), we have

$$\left|\alpha_i - \frac{P_{N,i}}{Q_{N,i}}\right| = \sum_{n \ge N+1} \frac{b^{f_i(n)}}{t^{f_i(n)} - b^{f_i(n)}} < 2\beta_i \left(\frac{b}{t}\right)^{f_i(N+1)} \le \frac{2\beta_i}{t^{Af_i(N+1)}}. \quad (8)$$

Now,

$$\begin{aligned}
Q_{N,1} \cdots Q_{N,k} &= \left(t^{f_1(N)} - b^{f_1(N)}\right) \cdots \left(t^{f_k(N)} - b^{f_k(N)}\right) \\
&= t^{f_1(N)+\cdots+f_k(N)} \left(1 - (b/t)^{f_1(N)}\right) \cdots \left(1 - (b/t)^{f_k(N)}\right).
\end{aligned}$$

For a fixed $E > 0$, since

$$G_{f_i}(N) := \frac{f_i(N+1)}{f_1(N) + \cdots + f_k(N)} \to \infty \quad (N \to \infty),$$

there exists $N_E \in \mathbb{N}$ such that if $N \geq N_E$, then

$$\begin{aligned} t^{Af_i(N+1)} &= t^{AG_{f_i}(N)(f_1(N)+\cdots+f_k(N))} \\ &\geq 2\beta_i \left(t^{f_1(N)} - b^{f_1(N)}\right)^E \cdots \left(t^{f_k(N)} - b^{f_k(N)}\right)^E \\ &= 2\beta_i \left(Q_{N,1} \cdots Q_{N,k}\right)^E, \end{aligned}$$

and so

$$\left|\alpha_i - \frac{P_{N,i}}{Q_{N,i}}\right| < \frac{1}{\left(Q_{N,1} \cdots Q_{N,k}\right)^E},$$

i.e., condition (iii) of Theorem 3 holds. The desired result follows now from Theorem 3.

3. Non-Liouville Lambert number

In contrast to the transcendental numbers appearing in part (ii) of Theorem 1 which are Liouville numbers, we show in this section that the transcendental numbers appearing in part (i) of the said theorem are not Liouville numbers in general.

The method we adopt is based on the analysis given in [5], which we now recall.

In [5, p. 476], Golomb showed that

$$G_1(z) := \sum_{n=1}^{\infty} \frac{z^{2^n}}{1 - z^{2^n}} = \sum_{n=1}^{\infty} i(n) z^n,$$

where $i(n)$ is the exponent of 2 in the prime decomposition of n, i.e., the number of zeros at the end of the binary representation of n. Consider the Lambert number (1) corresponding to $b = 1$, $t = 2$, $f(n) = 2^n$ in part (i) of Theorem 1, i.e.,

$$\begin{aligned} G_1\left(\frac{1}{2}\right) &= \sum_{n=1}^{\infty} \frac{1}{2^{2^n} - 1} \\ &= \sum_{n=1}^{\infty} \frac{i(n)}{2^n} = \frac{i(1)}{2} + \frac{i(2)}{2^2} + \frac{i(3)}{2^3} + \frac{i(4)}{2^4} + \frac{i(5)}{2^5} + \cdots \\ &:= 0.i(1)\, i(2)\, i(3)\, i(4)\, i(5) \ldots \\ &= 0.0102010301020104 \ldots; \end{aligned}$$

this last (base 2)-representation is improper in that $i(n)$ will sometimes exceed 1. We group the improper representation of $\sum_{n=1}^{\infty} i(n)/2^n$ into

segments, as follows:

$$\begin{aligned}
&0.(0)(1)(02)(0103)(01020104)(0102010301020105)\dots\\
&\quad=\left(\frac{0}{2}\right)+\left(\frac{1}{2^2}\right)+\left(\frac{0}{2^3}+\frac{2}{2^4}\right)+\left(\frac{0}{2^5}+\frac{1}{2^6}+\frac{0}{2^7}+\frac{3}{2^8}\right)\\
&\quad+\left(\frac{0}{2^9}+\frac{1}{2^{10}}+\frac{0}{2^{11}}+\frac{2}{2^{12}}+\frac{0}{2^{13}}+\frac{1}{2^{14}}+\frac{0}{2^{15}}+\frac{4}{2^{16}}\right)\\
&\quad+\left(\frac{0}{2^{17}}+\frac{1}{2^{18}}+\frac{0}{2^{19}}+\frac{2}{2^{20}}+\frac{0}{2^{21}}+\frac{1}{2^{22}}+\frac{0}{2^{23}}+\frac{3}{2^{24}}\right.\\
&\qquad\left.+\frac{0}{2^{25}}+\frac{1}{2^{26}}+\frac{0}{2^{27}}+\frac{2}{2^{28}}+\frac{0}{2^{29}}+\frac{1}{2^{30}}+\frac{0}{2^{31}}+\frac{5}{2^{32}}\right)\\
&\quad+\cdots. \qquad (9)
\end{aligned}$$

The 0th segment is (0), the 1st segment is (1), the 2nd segment is (02), the 3rd segment is (0103), etc. In general, the kth segment ($k>0$) is

$$\sum_{n=2^{k-1}+1}^{2^k}\frac{i(n)}{2^n},$$

has 2^{k-1} terms, and ends in k. For $k>0$, the average of all the terms in the kth segment is 1, and the running average, starting from the left of the segment is <1 until the very end. This means, for $k>1$, that the kth segment is numerically less that

$$\sum_{n=2^{k-1}+1}^{2^k}\frac{1}{2^n},$$

though of course positive.

To show that $G_1(1/2)$ is not a Liouville number, let $\omega>0$ and let p/q, where $p,q\in\mathbb{N}$, $q>1$, be any rational approximation of $G_1(1/2)$ such that

$$\left|G_1\left(\frac{1}{2}\right)-\frac{p}{q}\right|<\frac{1}{q^\omega}. \qquad (10)$$

Writing p/q uniquely as a base 2-representation, we see that rational numbers of the form $\frac{N}{2^Q}$, where $N,Q\in\mathbb{N}$, are dense in $[0,1]$ and so we assume that the denominator is of the form $q=2^Q$ ($Q\in\mathbb{N}$). Let Q correspond to the exponent in the k segment, i.e.,

$$2^{k-1}+1\le Q\le 2^k. \qquad (11)$$

Comparing the approximation in (10) with the representation in (9), we see that the base 2-representation of p/q must agree with that in (9) up

to the kth segment. As mentioned in [5, p. 477], the difference in (10) can be computed segment by segment, without carrying or borrowing between segments. Thus,

$$\begin{aligned}\frac{1}{2^{Q\omega}} = \frac{1}{q^{\omega}} &> \left| G_1\left(\frac{1}{2}\right) - \frac{p}{q} \right| \\ &\geq \text{sum of the form (9) starting from the } (k+1)\text{th segment onwards} \\ &\geq \frac{1}{2^{2^k+1}}.\end{aligned}$$

Using (11), we get

$$2^{2^k+1} > 2^{Q\omega} \geq 2^{(2^{k-1}+1)\omega}, \quad \text{i.e.,} \quad \omega < \frac{2^k+1}{2^{k-1}+1} < 2,$$

which is a contradiction if we choose, e.g., $\omega = 2$. Hence, $G_1(1/2)$ is a transcendental number which is most probably not a Liouville number.

As an application, consider

$$F(z) := \sum_{n=0}^{\infty} \frac{z^{2^n}}{1+z^{2^n}}.$$

Since [5, p. 476]

$$\frac{z}{1-z} = F(z) + G_1(z),$$

using the above result, we deduce that $F(1/2)$, the sum of the reciprocals of Fermat numbers, is a transcendental number which is most probably not a Liouville number.

4. Examples

In this section we provide examples illustrating several applications of Theorems 1 and 4.

1. Let $f_k : \mathbb{N} \to \mathbb{N}$ be defined by

$$f_k(n) = k^n \quad (k \in \mathbb{N},\ k > 2).$$

This function clearly satisfies the conditions of Theorem 1, and here

$$g_k(n) := \frac{f_k(n+1)}{f_k(n)} = k > 2.$$

Let A_k be a real number for which $2/k < A_k < 1$, and let $b, t \in \mathbb{N}$ such that $1 < b < t^{1-A_k}$. By Theorem 1 (i), the real number

$$\sum_{n\geq 0} \frac{b^{k^n}}{t^{k^n} - b^{k^n}}$$

is transcendental.

2. Let $f : \mathbb{N} \to \mathbb{N}$ be defined by

$$f(n) = n!\,.$$

This function clearly satisfies the conditions of Theorem 1, and here

$$g(n) := \frac{f(n+1)}{f(n)} = n + 1 \to \infty \quad (n \to \infty).$$

For a real number $A \in (0,1)$, let numbers $b, t \in \mathbb{N}$ be such that $1 < b < t^{1-A}$. By Theorem 1 (ii), the real number

$$\sum_{n\geq 0} \frac{b^{n!}}{t^{n!} - b^{n!}}$$

is a Liouville transcendental number.

3. Let $f_1, f_2 : \mathbb{N} \to \mathbb{N}$ be defined by

$$f_1(n) = n^{n!}, \qquad f_2(n) = (n-1)^{n!}.$$

These two functions satisfy the conditions of Theorem 4 with

$$\begin{aligned} g_1(n) &:= \frac{f_1(n+1)}{f_1(n) + f_2(n)} = \frac{(n+1)^{(n+1)!}}{n^{n!} + (n-1)^{n!}} \\ &\geq \lim_{n\to\infty} \frac{((n+1)^{n+1})^{n!}}{2(n+1)^{n!}} \to \infty \quad (n \to \infty), \\ g_2(n) &:= \frac{f_2(n+1)}{f_1(n) + f_2(n)} = \frac{n^{(n+1)!}}{n^{n!} + (n-1)^{n!}} \\ &\geq \lim_{n\to\infty} \frac{(n^{n+1})^{n!}}{2n^{n!}} \to \infty \quad (n \to \infty). \end{aligned}$$

Clearly, the function

$$\begin{aligned} h_n := f_1(n) - f_2(n) &= n^{n!}\left(1 - \left(1 - \frac{1}{n}\right)^{n!}\right) \\ &\to \infty \quad (n \to \infty). \end{aligned}$$

For $A \in (0,1)$, choose numbers $b, t \in \mathbb{N}$ such that $0 < b < t^{1-A}$. By Theorem 4, the two real numbers

$$\sum_{n\geq 0} \frac{b^{n^{n!}}}{t^{n^{n!}} - b^{n^{n!}}}, \qquad \sum_{n\geq 0} \frac{b^{(n-1)^{n!}}}{t^{(n-1)^{n!}} - b^{(n-1)^{n!}}}$$

are algebraically independent over $\mathbb{Q}$.

Acknowledgment

The origin of this work began when we were browsing through the list of publications of W. Schwarz in the book "From Arithmetic to Zeta-Functions. Number Theory in Memory of Wolfgang Schwarz", edited by J. Sander, J. Steuding and R. Steuding, Springer, 2016.

We have learned of Schwarz's passing from Prof. J. Steuding in September 2018 during the Sixth International Conference on Analytic Number Theory and Spatial Tessellations in Ukraine.

This article is dedicated to the memory of this beloved mathematician. Though, both authors have never met him, but his pioneer works touch us greatly.

References

1. W. W. Adams, *The algebraic independence of certain Liouville continued fractions*, Proc. Amer. Math. Soc. **95** (1985), 512–516.
2. Y. Bugeaud, *Approximation by Algebraic Numbers*, Cambridge Tracts in Mathematics No. 160, Cambridge University Press, 2004.
3. S. Chowla, *On series of the Lambert type which assume irrational values for rational values of the argument*, Proc. Nat. Inst. Sci. India **13** (1947), 171–173.
4. P. Erdös, *On arithmetical properties of Lambert series*, Proc. Nat. Inst. of Sciences of India **13** (1947), 63–66.
5. S. W. Golomb, *On the sum of the reciprocals of the Fermat numbers and related irrationalities*, Canad. J. Math. **15** (1963), 475–478.
6. W. M. Schmidt, *Diophantine Approximations*, Lecture Notes in Mathematics No. 785, Springer, Berlin, 1996.
7. W. Schwarz, *Remarks on the irrationality and transcendence of certain function*, Math. Scand. **20** (1967), 269–274.

Department of Mathematics, Faculty of Science, Kasetsart University, Bangkok, 10900, Thailand
Email: fscivil@ku.ac.th,angkana.s@ku.th

Voronoï's Impact on Modern Science, Book 6, Volume 1
Proceedings of the Sixth International Conference on Analytic Number Theory and Spatial Tessellations

DIRICHLET DIVISOR PROBLEM ON GAUSSIAN INTEGERS

Andrew V. Lelechenko (*Ukraine*)

Abstract. We improve existing estimates of moments of the Riemann zeta function. As a consequence, we are able to derive new estimates for the asymptotic behaviour of $\sum_{N\alpha\leq x} \mathfrak{t}_k(\alpha)$, where N stands for the norm of a complex number and $\mathfrak{t}_k$ is the k-dimensional divisor function on Gaussian integers.

Introduction

Define the divisor function τ and its generalisation, the k-dimensional divisor function τ_k, as follows:

$$\tau(n) = \sum_{\substack{d|n\\ d>0}} 1, \qquad \tau_k(n) = \sum_{\substack{d_1\cdots d_k=n\\ d_1,\ldots,d_k>0}} 1,$$

where n is a non-zero integer. That said, $\tau \equiv \tau_2$.

One can study asymptotic properties of summatory functions

$$\sum_{0<n\leq x} \tau(n) \qquad \text{and} \qquad \sum_{0<n\leq x} \tau_k(n).$$

It can be shown (see [8, (6.27)]) that for growing $x \to \infty$ we have

$$\sum_{0<n\leq x} \tau(n) = x\ln x + (2\gamma-1)x + O(x^{\theta_2}),$$
$$\sum_{0<n\leq x} \tau_k(n) = xP_k(\ln x) + O(x^{\theta_k}),$$

where $\gamma \approx 0.577$ denotes the Euler–Mascheroni constant, $P_k(y)$ is a fixed univariate polynomial of degree $k-1$, and θ_k is a positive real. The quest of deriving precise estimates for θ_k is one of the central problems of multiplicative number theory, known as Dirichlet divisor problem. It has been

Received August 15, 2018; revised November 21, 2018.
2010 *Mathematics Subject Classification.* 11A25, 11M06, 11N37, 11R11, 11Y70.
Key words and phrases. Riemann zeta function, exponent pairs, divisor function, Gaussian integers.

widely studied by various authors: Dirichlet has proved that $\theta_2 \le 1/2$, Voronoï improved this estimate to $\theta_2 \le 1/3+\varepsilon$ [14, 15], and more modern developments can be found in [6, 7, 13]. Namely,

$$\theta_2 \le 131/416, \qquad \theta_3 \le 43/96, \qquad \theta_k \le \frac{k-1}{k+2}, \quad k \ge 4.$$

It is natural to extend the notion of divisor functions from integers to other unique factorisation domains such as rings of quadratic integers. Let R be a principal ring of imaginary quadratic integers. Then it is possible to define divisor functions $\mathfrak{t}, \mathfrak{t}_k \colon R \setminus 0 \to \mathbb{N}$ as

$$\mathfrak{t}(\alpha) = \sum_{\substack{d \mid \alpha \\ d \in R/U(R)}} 1, \qquad \mathfrak{t}_k(\alpha) = \sum_{\substack{d_1 \cdots d_k \sim \alpha \\ d_1, \dots, d_k \in R/U(R)}} 1,$$

where $\alpha \sim \beta$ iff $\alpha \mid \beta$ and $\beta \mid \alpha$, and $U(R)$ is a group of units of R.

Further, if R is also a Euclidean ring equipped with norm $N \colon R \to \mathbb{N}$, one can learn asymptotic properties of $\sum\limits_{0<N\alpha\le x} \mathfrak{t}_k(\alpha)$. Lai Dyk Thin [12] has proved that

$$\sum_{0<N\alpha\le x} \mathfrak{t}(\alpha) = l_1 x \ln x + l_2 x + O(x^{3/5+\varepsilon}), \tag{1}$$

$$\sum_{0<N\alpha\le x} \mathfrak{t}_k(\alpha) = x L_k(\ln x) + O(x^{1-1/(k+1)+\varepsilon}), \tag{2}$$

where constants l_1, l_2 and polynomial L_k of degree $k-1$ depend only on R.

The aim of following notes is to improve the error term in (1) and (2) in the specific case of Gaussian integers $\mathbb{Z}[i]$. Namely, from now on

$$\mathfrak{t}(\alpha) = \sum_{\substack{d \mid \alpha \\ \arg d \in [0,\pi/2)}} 1, \qquad \mathfrak{t}_k(\alpha) = \sum_{\substack{d_1 \cdots d_k \sim \alpha \\ \arg d_1, \dots, \arg d_k \in [0,\pi/2)}} 1.$$

and $N(a+bi) = a^2+b^2$. In order to obtain this result we improve known estimates of moments of the Riemann zeta function.

1. Moments of ζ

As usual $\zeta(s)$ is the Riemann zeta function. Real and imaginary components of the complex s are denoted as $\sigma := \Re s$ and $t := \Im s$, so $s = \sigma + it$.

Denote by $M(A)$ a real function such that

$$\int_1^T |\zeta(1/2+it)|^A \, dt \ll T^{M(A)+\varepsilon}. \tag{3}$$

Estimates of the Riemann zeta function on critical line $\sigma = 1/2$ are crucial for many applications. The best known result can be found in [5, Th. 8.3]. The following theorem improves it for $A > 12$.

Theorem 1. *The following choice of M is valid and satisfies* (3)*:*

$$M(A) = \begin{cases} 1 + \dfrac{A-4}{8}, & 4 \le A \le 12, \\ 1 + \max\left\{ \dfrac{13}{84}(A-6), E(A) \right\}, & 12 < A \le C, \\ 1 + \dfrac{13}{84}(A-6), & C < A, \end{cases} \tag{4}$$

where

$$E(A) = \inf_{(k,l)\in P} \left\{ \frac{l}{k} \;\middle|\; (4-A)k + 4l + 2 \ge 0 \right\}, \tag{5}$$

$$C = \inf_{(k,l)\in P} \left\{ \frac{4k+4l+2}{k} \;\middle|\; 1 - k - \frac{16}{13}l \ge 0 \right\} = \frac{16645467}{972266} = 17.1\ldots \tag{6}$$

and P denotes the set of exponent pairs, defined in accordance to Krätzel [8, Ch. 2]*.*

Proof. The first case follows from the estimates

$$\int_1^T |\zeta(1/2+it)|^4\,dt \ll T\log^4 T$$

by Ingham [4] and

$$\int_1^T |\zeta(1/2+it)|^{12}\,dt \ll T^2\log^{17} T$$

by Heath-Brown [3]. Precisely, let us denote $\xi(t) = |\zeta(1/2+it)|$. By Hölder inequality we have

$$\int_1^T \xi^A(t)\,dt = \int_1^T \xi^{(12-A)/2}(t)\xi^{(3A-12)/2}(t)\,dt \ll$$
$$\ll \left(\int_1^T \xi^{q_1(12-A)/2}(t)\,dt\right)^{1/q_1} \left(\int_1^T \xi^{q_2(3A-12)/2}(t)\,dt\right)^{1/q_2}$$

for $1/q_1 + 1/q_2 = 1$. Taking $q_1 = 8/(12-A)$ and $q_2 = 8/(A-4)$ we get

$$\int_1^T \zeta^A(1/2+it)\,dt \ll$$
$$\ll \left(\int_1^T \xi^4(t)\,dt\right)^{(12-A)/8} \left(\int_1^T \xi^{12}(t)\,dt\right)^{(A-4)/8} \ll$$
$$\ll T^{(12-A)/8}\log^{4(12-A)/8} T \cdot T^{2(A-4)/8}\log^{17(A-4)/8} T =$$
$$= T^{1+(A-4)/8}\log^{(13A-20)/8} T.$$

Consider the second case. Denoting R and V as in Ivić [5, (8.6)]. It is enough to show that for $A > 12$ and for every exponent pair $(k,l) \in P$ such that $(4-A)k + 4l + 2 \geq 0$ we have

$$R \ll T^{1+\max\{\frac{13}{84}(A-6),\,l/k\}+\varepsilon} V^{-A}. \tag{7}$$

But by [5, Th. 8.2] we have

$$R \ll T^{1+\varepsilon}V^{-6} + T^{1+l/k+\varepsilon}V^{-2(1+2k+2l)/k}.$$

By condition on (k,l) we know that $V^{-2(1+2k+2l)/k} \ll V^{-A}$, so

$$R \ll T^{1+\varepsilon}V^{-A}(V^{A-6} + T^{l/k}).$$

But definitely since $\zeta(1/2+it) \ll t^{13/84+\varepsilon}$ by Bourgain [1] we have $V \ll T^{13/84+\varepsilon}$, which completes the proof of (7).

Now let us investigate the third case. Let $A > C$. Then by definition of C there is an exponent pair (k,l) such that $1 - k - 16l/13 \geq 0$ and $(4k+4l+2)/k < A$. So by [5, Th. 8.2] we have

$$R \ll T^{1+\varepsilon}V^{-6}(1 + T^{l/k}V^{-(-2k+4l+2)/k}) \ll$$
$$\ll \begin{cases} T^{1+\varepsilon}V^{-6}, & V \gg T^c, \\ T^{1+l/k+\varepsilon}V^{-(4k+4l+2)/k}, & \text{otherwise}, \end{cases}$$

where $c = l/(-2k+4l+2)$. One can check that condition

$$1 - k - 16l/13 \geq 0$$

implies $c \leq 13/84$.

It is enough to prove that

$$S := \sum_{r \leq R} |\zeta(1/2+it_r)|^A \ll T^{1+\frac{13}{84}(A-6)+\varepsilon},$$

where $|t_r| \leq T$, $|t_r - t_s| \geq 1$ for $1 \leq r \neq s \leq R$. Cf. [5, (8.58)].

Again since $\zeta(1/2+it) \ll t^{13/84+\varepsilon}$ we can split $\{t_r\}_{r=1}^{R}$ into $1+\lfloor \frac{13}{84}\log T\rfloor$ disjoint subsets $\{t_{j,r_j}\}_{r_j=1}^{R_j}$, where for every t_{j,r_j} we have

$$V_j := 2^j \le |\zeta(1/2+it_{j,r_j})| \le 2^{j+1}.$$

Then

$$S \ll \sum_{j=0}^{\lfloor \frac{13}{84}\log T\rfloor} \sum_{r\le R_j} 2^{Aj} \ll \left(\sum_{j=0}^{c\log T} + \sum_{j=c\log T}^{\lfloor \frac{13}{84}\log T\rfloor}\right) R_j 2^{Aj} =: S_2 + S_1.$$

Here by choice of c for $j \ge c\log T$ we have $R_j \ll T^{1+\varepsilon}2^{-6j}$, so

$$S_1 \ll T^{1+\varepsilon} \sum_{j=c\log T}^{\lfloor \frac{13}{84}\log T\rfloor} 2^{(A-6)j} \ll T^{1+\frac{13}{84}(A-6)+\varepsilon}.$$

On the other side for $j \le c\log T$ we have $R_j \ll T^{1+l/k+\varepsilon}2^{-(4k+4l+2)j/k}$, so

$$S_2 \ll T^{1+l/k+\varepsilon} \sum_{j=0}^{c\log T} 2^{(A-(4k+4l+2)/k)j} \ll T^{1+l/k+(A-(4k+4l+2)/k)c+\varepsilon}.$$

But

$$l/k + (A-(4k+4l+2)/k)c = l/k + (A-6-(-2k+4l+2)/k)c =$$
$$= l/k + (A-6)c - l/k = (A-6)c \le \frac{13}{84}(A-6),$$

which completes the proof. □

To apply Theorem 1 efficiently, we need a method to evaluate infimums of form (5) and (6) over exponent pairs. We have developed such framework, whose initial version has been described in [9]. Since then the framework has been developed further and released as a `exp-pairs` package [10]. For instance, an estimate of C, given in (6), corresponds to the choice

$$(k,l) = BA(A(BAA)^2)^2A^2(BA)^2ABA(13/84, 55/84),$$

where A and B stand for application of A- and B-process from Krätzel [8, Ch. 2], and $(13/84, 55/84)$ is an exponent pair by Bourgain [1].

We computed Table 1 as a reference for estimates of $M(A)$ provided by Theorem 1 for integer $A \in (12, C)$.

TABLE 1. Values of $M(A)$ for $A = 13, 14, 15, 16, 17$

A	$M(A)$	Exponent pair for $E(A)$
13	2.134766...	$BAA^3(BA)^2A^2BAA(BA)^5A^5BA(A(BA)^2A)^2 \cdot$ $\cdot BA((BA)^2A)^2(ABA)^3BAA^4(BA)^7A^4BAA^5\cdots$
14	$\frac{1117297289}{491431296}$	$BAA^2BA(BAA)^5((BA)^2A)^2A(BA)^3A^3 \cdot$ $\cdot (ABA)^2(1/6, 2/3)$
15	$\frac{61902400787}{25629743097}$	$(BAA^2)^2(BA)^2A^4(BA)^4ABA(A(BA)^3A)^2 \cdot$ $\cdot (BA)^5A(1/6, 2/3)$
16	2.558254...	$BAA\big((BA)^2ABAA(BA)^2A^2(BA)^4\big)^\infty$
17	$\frac{7321}{2708}$	$(BAA)^3A^2(13/84, 55/84)$

2. MOMENTS OF Z

Let us briefly recap key properties of Gaussian integers. The ring $\mathbb{Z}[i]$ consists of $\alpha = a + bi$ for integer a and b.

There are four units of the ring: $U(\mathbb{Z}[i]) = \{1, i, -1, -i\}$. For any $\alpha \neq 0$ its orbit under action of the unit group consists of four elements, one per each quadrant. We will use an element of the orbit from the first quadrant (such that $\arg\alpha \in [0, \pi/2]$) as a canonical representative.

The ring is equipped with norm $N(a+bi) = a^2 + b^2$, which is a homomorphism of multiplicative group: $N(\alpha \cdot \beta) = N(\alpha) \cdot N(\beta)$. The ring is Euclidean and principal, so it is a unique factorisation domain.

Gaussian integer $\mathfrak{p}$ is prime if and only if one of the following cases has place:

- $\mathfrak{p} \sim 1 + i$,
- $\mathfrak{p} \sim p$, where $p \equiv 3 \pmod 4$,
- $N(\mathfrak{p}) = p$, where $p \equiv 1 \pmod 4$.

In the last case there are exactly two non-associated $\mathfrak{p}_1$ and $\mathfrak{p}_2$ such that $N(\mathfrak{p}_1) = N(\mathfrak{p}_2) = p$. See [2, §34].

Let $Z(s)$ denote the Dedekind zeta function of $\mathbb{Z}[i]$, which is a Gaussian analogue of the Riemann zeta function. Namely,

$$Z(s) = \sum_{\substack{\alpha \neq 0 \\ \arg\alpha \in [0, \pi/2)}} N(\alpha)^{-s}.$$

Let β be the Dirichlet beta function,

$$\beta(s) = \sum_{n\geq 0} \frac{(-1)^n}{(2n+1)^{-s}}.$$

Converting Dirichlet sums to Euler products and back, we have

$$\begin{aligned} Z(s) = &\prod_{\substack{\mathfrak{p} \\ \arg \mathfrak{p} \in [0,\pi/2)}} (1-N(\mathfrak{p})^{-s})^{-1} = \\ &= \frac{1}{1-2^{-s}} \prod_{p\equiv 3 \pmod 4} \frac{1}{1-p^{-2s}} \prod_{p\equiv 1 \pmod 4} \frac{1}{(1-p^{-s})^2} = \\ &= \prod_{p} \frac{1}{1-p^{-s}} \prod_{p\equiv 3 \pmod 4} \frac{1}{1+p^{-s}} \prod_{p\equiv 1 \pmod 4} \frac{1}{1-p^{-s}} = \\ &= \zeta(s)\beta(s). \end{aligned}$$

Denote by $I(A)$ a real function such that

$$\int_1^T |Z(1/2+it)|^A \, dt \ll T^{I(A)+\varepsilon}.$$

Firstly, $I(2) = 1$, because

$$\begin{aligned} \int_1^T |Z(1/2+it)|^2 \, dt \ll \\ \ll \left(\int_1^T |\zeta(1/2+it)|^4 \, dt \right)^{1/2} \left(\int_1^T |\beta(1/2+it)|^4 \, dt \right)^{1/2} \ll \\ \ll T^{(1+\varepsilon)/2} T^{(1/2+\varepsilon)/2} = T^{1+\varepsilon}, \end{aligned}$$

where we applied estimates for fourth moments from Montgomery [11, Th. 10.1].

Secondly, $I(3) = 5/4$, because

$$\begin{aligned} \int_1^T |Z(1/2+it)|^3 \, dt \ll \\ \ll \left(\int_1^T |\zeta(1/2+it)|^{12} \, dt \right)^{1/4} \left(\int_1^T |\beta(1/2+it)|^4 \, dt \right)^{3/4} \ll \\ \ll T^{(2+\varepsilon)/4} T^{3(1+\varepsilon)/4} = T^{5/4+\varepsilon}, \end{aligned}$$

where the estimate for the twelfth moment is by Ingham [4].

Further, estimates for higher moments of Dedekind zeta function are given by the following theorems.

Theorem 2.

$$I(A) = \begin{cases} \left(\frac{13}{84} + \frac{M(D) - \frac{8}{21}}{D}\right) A + \frac{8}{21} \\ \qquad = 0.2896A + \frac{8}{21}, & 4 \le A \le D, \\ \frac{13}{84}A + M(A), & D \le A. \end{cases} \tag{8}$$

where $D = 12.4868\ldots$.

Proof. For brevity below Z^A means $|Z(1/2+it)|^A$, ζ^A means $|\zeta(1/2+it)|^A$, β^A means $|\beta(1/2+it)|^A$, and $\int \cdot$ stands for $\int_1^T \cdot\, dt$.

We start with the first case. Let D be any real greater than A. Then for $b = 4(D-A)/D$ we write

$$\int Z^A = \int \zeta^A \beta^A = \int \zeta^A \beta^b \beta^{A-b}.$$

Since $\beta \ll T^{13/84+\varepsilon}$ for $t \in [1, T]$ we get

$$\int Z^A \ll T^{13(A-b)/84+\varepsilon} \int \zeta^A \beta^b. \tag{9}$$

Now apply Hölder inequality with

$$q_1 = D/A, \quad q_2 = D/(D-A), \quad 1/q_1 + 1/q_2 = 1$$

to obtain

$$\int \zeta^A \beta^b \ll \left(\int \zeta^D\right)^{A/D} \left(\int \beta^4\right)^{(D-A)/D} \ll \\ \ll T^{AM(D)/D+\varepsilon} T^{(D-A)/D+\varepsilon}. \tag{10}$$

Combination of (9) and (10) provides us with the first statement of the theorem.

It remains to choose D to minimize $(M(D) - \frac{8}{21})/D$ and numerical computations by Theorem 1 give us $D = 12.4868\ldots$

The second statement is almost trivial, since

$$\int Z^A = \int \zeta^A \beta^A \ll T^{13A/84+\varepsilon} \int \zeta^A \ll T^{13A/84+M(A)+\varepsilon}. \qquad \square$$

3. Summatory function of $\mathfrak{t}_k$

Now we are ready to attack our main aim: the summatory function of $\mathfrak{t}_k$. Just to get better acquainted we refer readers to Table 2 for the plot of values of $\mathfrak{t}$.

TABLE 2. Values of $\mathfrak{t}(a+bi)$ in the first quadrant. Here a increases horizontally and b vertically. Numbers with exactly two divisors are Gaussian primes.

15	4	2	8	2	12	4	4	3	8	8	4	4	4	2	16
14	2	12	4	6	4	8	4	12	2	8	2	12	4	8	2
13	8	2	4	4	4	4	4	2	8	2	8	2	8	4	4
12	4	6	4	12	3	12	2	10	6	6	4	12	2	12	4
11	4	4	8	2	4	2	8	4	4	4	4	4	8	2	4
10	2	8	2	6	6	8	2	6	2	16	4	6	2	8	8
9	4	4	8	2	4	4	8	4	6	2	4	6	8	2	8
8	4	6	2	10	2	9	2	8	4	6	4	10	2	12	3
7	6	2	4	4	4	4	4	2	8	2	8	2	4	4	4
6	2	8	4	6	2	8	4	9	4	8	2	12	4	8	4
5	4	2	4	2	8	2	4	2	4	6	4	3	4	4	12
4	2	6	3	6	2	6	4	10	2	6	2	12	4	6	2
3	4	2	4	3	4	4	4	2	8	2	8	4	4	4	8
2	2	4	2	6	2	8	2	6	4	8	4	6	2	12	2
1	2	2	4	2	4	2	6	4	4	2	4	4	8	2	4
0	1	3	2	5	4	6	2	7	3	12	2	10	4	6	8
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Similar to the real case $\sum\limits_{n>0} \tau_k(n) n^{-s} = \zeta^k(s)$, one can check that

$$\sum_{\substack{\alpha \neq 0 \\ \arg \alpha \in [0, \pi/2)}} \mathfrak{t}_k(\alpha) N(\alpha)^{-s} = Z^k(s).$$

Theorem 3. *Let* $S_k(x) = \sum\limits_{\substack{0 < N\alpha \leq x \\ \arg \alpha \in [0, \pi/2)}} \mathfrak{t}_k(\alpha)$. *Then*

$$\begin{aligned} S_2(x) &= l_1 x \ln x + l_2 x + O(x^{1/2+\varepsilon}), \\ S_3(x) &= x P_3(\log x) + O(x^{3/5+\varepsilon}), \\ S_k(x) &= x P_k(\log x) + O(x^{1-1/2I(k)+\varepsilon}). \end{aligned} \tag{11}$$

where

$$\begin{aligned} l_1 &= \pi^2/16 = 0.61685\ldots, \\ l_2 &= \pi^2(2\gamma - 1)/16 + \pi\beta'(1)/2 = 0.39827\ldots, \\ \deg P_k &= k - 1. \end{aligned}$$

Proof. By Perron formula we have

$$S_k(x) = \operatorname*{res}_{s=1} Z^k(s) x^s / s + O(xT^{-1} + x^{1/2} T^{I(k)-1+\varepsilon}).$$

TABLE 3. Values of $S(x) = \sum_{\substack{0<N\alpha\le x \\ \arg\alpha\in[0,\pi/2)}} \mathfrak{t}(\alpha)$ and estimates $\bar{S}(x)$ by (11)

x	$S(x)$	$\bar{S}(x)$	$\frac{S(x)-\bar{S}(x)}{\sqrt{x}}$
10	24	18	1.84
100	337	324	1.31
1000	4694	4659	1.10
10000	60857	60797	0.60
100000	750259	750002	0.81
1000000	8920571	8920371	0.20
10000000	103407407	103407214	0.06
100000000	1176104936	1176107167	−0.22
1000000000	13181419972	13181421920	−0.06

Here the residues give the main term of form $xP_k(\log x)$. Let us analyse the case of $k = 2$ in details. We have

$$\begin{aligned}\operatorname*{res}_{s=1} Z^2(s)x^s/s &= \operatorname*{res}_{s=1}(\zeta^2(s)/s\cdot\beta^2(s)\cdot x^s) = \\ &= \operatorname*{res}_{s=1}\left(\frac{1}{(s-1)^2}+\frac{2\gamma-1}{s-1}+O(1)\right)\times \\ &\times\left(\frac{\pi}{4}+\frac{\pi\beta'(1)}{2}(s-1)+O(s-1)^2\right)^2\times \\ &\times\left(x+x\log x(s-1)+O(s-1)^2\right) = \\ &= \frac{\pi^2}{16}x\ln x+\left(\frac{\pi^2(2\gamma-1)}{16}+\frac{\pi\beta'(1)}{2}\right)x.\end{aligned}$$

With regards to the error term let us choose $T = x^a$ in order to minimize the magnitude of $O(xT^{-1} + x^{1/2}T^{I(k)-1+\varepsilon})$. One can check that $T = x^{1/2I(k)}$ turns this expression into $O(x^{1-1/2I(k)+\varepsilon})$.

Error terms for $k = 2$ and $k = 3$ are consequences of estimates for $I(2)$ and $I(3)$ obtained in the previous section. □

Theorem 3 improves results of Thin (1) and (2) for all k. Cases $k = 2$ and $k = 3$ were given above. For $4 \le k \le 12$ by (4) and (8) we have

$$2I(k) \le 0.2896k + 0.3806 < k < k + 1.$$

For $13 \le k$ we have

$$2I(k) \le 2(1 + 13/84(2k - 6)) = 52/84k + 1/7 < k < k + 1.$$

It is interesting to check the accuracy of our asymptotic estimate on some numerical data. Table 3 shows values of summatory function for growing x, compared against predictions from (11). The last column lists error terms divided by $x^{1/2}$ and fuels our confidence in the correct order of O-bound.

References

1. J. Bourgain, *Decoupling, exponential sums and the Riemann zeta function*, J. Amer. Math. Soc. **30** (2017), 205–224.
2. C. F. Gauss, *Theoria residuorum biquadraticorum, commentatio secunda*, Commentationes soc. reg. sc. Gotting. recentiores **7** (1832).
3. D. R. Heath-Brown, *The twelfth power moment of the Riemann-function*, Quart. J. Math. Oxford II. Ser. **29** (1978), 443–462.
4. A. E. Ingham, *Mean-value theorems in the theory of the Riemann zeta-function*, Proc. Lond. Math. Soc. **27** (1926), no. 2, 273–300.
5. A. Ivić, *The Riemann zeta-function: Theory and applications*, Dover Publications, Mineola, New York, 2003.
6. A. Ivić and M. Ouellet, *Some new estimates in the Dirichlet divisor problem*, Acta Arith. **52** (1989), no. 3, 241–253.
7. G. Kolesnik, *On the estimation of multiple exponential sums*, Recent progress in analytic number theory, vol. 1, Academic Press, London, 1981, pp. 231–246.
8. E. Krätzel, *Lattice points*, Kluwer, Dordrecht, 1988.
9. A. V. Lelechenko, *Linear programming over exponent pairs*, Acta Univ. Sapientiae, Inform. **5** (2013), no. 2, 271–287.
10. A. V. Lelechenko, *Exp-pairs: linear programming over exponent pairs*, `http://hackage.haskell.org/package/exp-pairs`, March 2018.
11. H. L. Montgomery, *Topics in multiplicative number theory*, vol. 227, Springer Verlag, 1971.
12. Laĭ Dyk Thin', *On the number of divisors in angles* (Russian), Mat. Sb., N. Ser. **67 (109)** (1965), no. 3, 345–365.
13. E. C. Titchmarsh, *The theory of the Riemann zeta-function*, 2nd, rev. ed., Oxford University Press, New-York, 1986.
14. G. Voronoï, *Sur un problème du calcul des fonctions asymptotiques*, J. Reine Angew. Math. **126** (1903), 241–282.
15. G. Voronoï, *Sur une fonction transcedente et ses applications à la sommation de quelques séries*, Ann. Sci. Éc. Norm. Supér. **21** (1903), 203–267.

Department of Computer Algebra and Discrete Mathematics, Odessa National University, 2 Dvoryanskaya St., Odessa, 65082, Ukraine
Email: `1@dxdy.ru`

Voronoï's Impact on Modern Science, Book 6, Volume 1
Proceedings of the Sixth International Conference on Analytic Number Theory and Spatial Tessellations

INTEGER SOLUTIONS OF IMPLICIT LINEAR DIFFERENCE EQUATIONS

Vsevolod Martseniuk, Sergiy Gefter, and Aleksey Piven' (*Ukraine*)

Abstract. We obtain some conditions of existence and uniqueness for solutions in integer numbers of implicit linear non-homogeneous difference equations with constant integer coefficients and integer non-homogeneities.

Introduction

Let c_0, c_1, c_2, …, c_k be integers and let $\{f_n\}$ be a given sequence of integers. In this work we study a problem of a solving in integer numbers for the following implicit linear difference equation of kth order:

$$c_k x_{n+k} = c_{k-1}x_{n+k-1} + c_{k-2}x_{n+k-2} + \cdots + c_0 x_n + f_n, \quad n = 0, 1, 2, \ldots. \quad (1)$$

When $c_k = \pm 1$, this equation has infinitely many solutions in integer numbers. If $c_k \neq \pm 1$ and $c_k \neq 0$, then Equation (1) is an implicit equation over the ring $\mathbb{Z}$.

Let note that an implicit equation could has not any integer solution. For example, the general solution of the equation $3x_{n+1} = x_n + 1$ over $\mathbb{Q}$ is of the form $x_n = \frac{a}{3^n} + \frac{1}{2}$, where $a \in \mathbb{Q}$, $n = 0, 1, 2, \ldots$. It is obvious that for any value of a constant a we can not obtain an integer solution (see also Corollary 3 and Example 7). In this paper we show that in a lot of cases if there exists an integer solution then this solution is unique (see Theorems 1–4). Furthermore we obtain a criterion of an existence of a solution in integer numbers for Equation (1) with a constant right-hand side (Theorem 5), and we consider some special cases and examples, which illustrate a constructed theory (see Sections 2, 3). In Section 4 we study a linear implicit difference equation of the first-order over the ring of p-adic integers.

Integer solutions of linear homogeneous difference equation were studied implicitly by Valerii Berestovskiĭ and Yurii Nikonorov in connection

Received July 31, 2018; revised November 21, 2018.
2010 *Mathematics Subject Classification.* Primary 11D72; Secondary 65Q10.
Key words and phrases. Difference equations, integer solution, characteristic polynomial.

with a consideration of Pisot numbers in the paper [1]. Solutions in integer numbers of some non-homogeneous first-order linear difference equations were studied in [4, 6].

1. Study of implicit linear difference equation of kth order

In this section we obtain a few uniqueness conditions for integer solutions of Equation (1). For the case, when $\{f_n\}$ is constant, from the uniqueness we obtain a criterion of existence of an integer solution for such equation (see Theorem 5).

Theorem 1. *Let the all complex roots of the characteristic polynomial*

$$c_k\lambda^k - (c_{k-1}\lambda^{k-1} + c_{k-2}\lambda^{k-2} + \cdots + c_0) \tag{2}$$

of Equation (1) *belong to the ring* $0 < |\lambda| < 1$. *Then the homogeneous difference equation*

$$c_k x_{n+k} = c_{k-1}x_{n+k-1} + c_{k-2}x_{n+k-2} + \cdots + c_0 x_n, \quad n = 0, 1, 2, \ldots, \tag{3}$$

has only trivial integer solution.

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_l$, $l \le k$, be the complex roots of the characteristic polynomial of Equation (1). Since $|\lambda_j| < 1$, we have $\lambda_j^n \to 0$, $j = 1, 2, \ldots, l$. Let now $\{x_n\}$ be an integer solution of the equation (3). Then $x_n = \sum\limits_{j=1}^{l} Q_j(n)\lambda_j^n$, where $Q_1, Q_2, \ldots, Q_l$ are polynomials. Hence $x_n \to 0$ in $\mathbb{C}$. Since x_n are integer, we obtain that there exists $n_0 \in \mathbb{N}$ such that $x_n = 0$ for $n \ge n_0$. From Equation (3) it follows that $x_n = 0$ for all $n = 0, 1, 2, \ldots$. □

Corollary 1. *Suppose that* $c_0 \neq 0$ *and the following inequality*

$$|c_k| > |c_{k-1}| + |c_{k-2}| + \cdots + |c_0| \tag{4}$$

is fulfilled. Then the homogeneous equation (3) *has only trivial integer solution.*

Proof. Since $c_0 \neq 0$, we have that $\lambda = 0$ is not a root of the characteristic polynomial (2). We will show that all the roots of the characteristic polynomial of Equation (3) satisfy the strict inequality $|\lambda| < 1$.

Suppose that we can found a root of this polynomial satisfying $|\lambda| \ge 1$. Then we have:

$$\begin{aligned} |c_k\lambda^k| &= |c_{k-1}\lambda^{k-1} + c_{k-2}\lambda^{k-2} + \cdots + c_0| \le \\ &\le |c_{k-1}\lambda^{k-1}| + |c_{k-2}\lambda^{k-2}| + \cdots + |c_0| \le \\ &\le |\lambda^k|(|c_{k-1}| + |c_{k-2}| + \cdots + |c_0|) \end{aligned}$$

since $|\lambda| \geq 1$. Therefore $|c_k| \leq |c_{k-1}| + |c_{k-2}| + \cdots + |c_0|$ and we obtain the contradiction. □

The following uniqueness theorem is based on properties of divisibility of coefficients of the equation.

Theorem 2. *Suppose c_k, c_{k-1}, c_{k-2}, …, c_1 have the common divisor p, but c_0 is not divisible by p. Then the homogeneous equation* (3) *has only trivial solution.*

Proof. Since

$$c_0 x_n = c_k x_{n+k} - c_{k-1} x_{n+k-1} - \cdots - c_1 x_{n+1},$$

we obtain that x_n is divisible by p for all $n = 0, 1, 2, 3, \ldots$. But now from (3) it follows that x_n is divisible by p^2 for $n = 0, 1, 2, 3, \ldots$. Repeating this reasoning we obtain that x_n is divisible by p^m for all $m = 1$, 2, 3, …. This means that $x_n = 0$. □

The following theorem gives a general sufficient condition of uniqueness. This theorem is, in fact, a variant of Theorem of Berestovskiĭ and Nikonorov (see [1, Theorem 6 in § 3 (Russian ed.) or Theorem 2 in Section 3 (English ed.)]).

Theorem 3. *Consider the implicit linear homogeneous difference equation* (3)*, where $c_j \in \mathbb{Z}$, $j = 0, 1, 2, \ldots, k$, $c_k \geq 2$, and not all the numbers c_{k-1}, c_{k-2}, …, c_0 are divisible by c_k. Assume that the characteristic polynomial* (2) *of Equation* (3) *is irreducible over $\mathbb{Q}$. Then Equation* (3) *has only trivial solution.*

Proof. Suppose that Equation (3) has a non-trivial integer solution. Write its characteristic equation in the following form:

$$\lambda^k - \frac{c_{k-1}}{c_k}\lambda^{k-1} - \frac{c_{k-2}}{c_k}\lambda^{k-2} - \cdots - \frac{c_1}{c_k}\lambda - \frac{c_0}{c_k} = 0. \tag{5}$$

The left-hand side of Equation (5) is irreducible over $\mathbb{Q}$ too. According to Theorem of Berestovskiĭ and Nikonorov [1] the all coefficients of Equation (5) are integers. We obtain the contradiction with the condition that not all the numbers c_{k-1}, c_{k-2}, …, c_0 are divisible by c_k. □

Corollary 2. *Suppose $a, b, c \in \mathbb{Z}$, $c \neq \pm 1$, the characteristic polynomial*

$$c\lambda^2 - b\lambda - a$$

of the difference equation

$$c x_{n+2} = b x_{n+1} + a x_n$$

has no rational roots, and a or b is not divisible by c. Then this equation has the only trivial integer solution.

The following sufficient condition of uniqueness is based on an application of some arithmetic properties of entire function of complex variable (see [2, 7]).

Theorem 4. *Consider the implicit linear homogeneous difference equation* (3)*, where* $c_j \in \mathbb{Z}$*,* $j = 0, 1, 2, \ldots, k$*, and* $c_k \geq 2$*. Suppose that all the roots of the characteristic polynomial* (2) *belong to the set* $(\frac{1}{2}, 1) \cup (1, 2)$*. Then Equation* (3) *has only trivial solution.*

Proof. Let all of the roots λ_1, λ_2, ..., λ_k of the characteristic polynomial (2) are different (general situation is considered analogously). Then the general solution of Equation (3) has the following form:

$$x_n = \sum_{j=1}^{k} \alpha_j \lambda_j^n, \quad n = 0, 1, 2, \ldots. \tag{6}$$

Consider the following entire function of complex variable

$$g(z) = \sum_{j=1}^{k} \alpha_j \lambda_j^z, \quad z \in \mathbb{C}. \tag{7}$$

Then $g(z)$ is a function of an exponential type, and its exponential type less than $\ln 2$ (see [2]). Moreover, $g(n) = x_n \in \mathbb{Z}$, $n = 0, 1, 2, \ldots$.

By Polya's theory on integer-valued entire functions, we have that $g(z)$ is polynomial (see [7, Section 1.1] and [2, Section 9.12]). Not so hard to show that it is possible only if $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$. Thus $g(z) = 0$. In particular, $x_n = g(n) = 0$ for all n. □

Examples in Section 3 show that the above-considered sufficient conditions of uniqueness supplement each other only.

The following statement is a general sufficient condition of existence of integer solutions.

Theorem 5. *Let* c_0, c_1, ..., $c_k \in \mathbb{Z}$ *and* $f_n = f \in \mathbb{Z}$ *for all* $n = 0$, 1, 2, ... *If the homogeneous equation* (3) *has only trivial solution, then* $c_k - (c_{k-1} + c_{k-2} + \cdots + c_0) \neq 0$ *and the non-homogeneous equation*

$$c_k x_{k+n} = c_{k-1} x_{k+n-1} + c_{k-2} x_{k+n-2} + \cdots + c_0 x_n + f, \quad n = 0, 1, 2, \ldots, \tag{8}$$

has an integer solution if and only if

$$c_k - (c_{k-1} + c_{k-2} + \cdots + c_0)$$

is a divisor of f*.*

Proof. Since the constant sequence $z_n = 1$ is not a solution of Equation (3), we obtain $c_k - c_{k-1} - \cdots - c_0 \neq 0$.

Let $\{x_n\}$ be an integer solution of Equation (8). Set $y_n = x_{n+1}$. Then the sequence $\{y_n\}$ is an integer solution of Equation (8) as well. Then $y_n = x_n$, i.e., $x_n = x_0$ for all n and $(c_k - c_{k-1} - c_{k-2} - \cdots - c_0)x_0 = f$.

Conversely, if $c_k - (c_{k-1} + c_{k-2} + \cdots + c_0)$ is a divisor of f, then the constant sequence

$$x_n = \frac{f}{c_k - c_{k-1} - c_{k-2} - \cdots - c_0}$$

is an integer solution of (8). □

2. The case of the first-order equation

Let us consider the implicit linear difference equation of the first order:

$$bx_{n+1} = ax_n + f_n, \quad n = 0, 1, 2, \ldots, \tag{9}$$

where a, b are integers, $a \neq 0$, $b \neq 0$, and $\{f_n\}$ is a sequence of integers. As opposed to the general case in this situation we can obtain the following simple criterion of uniqueness.

Theorem 6. *The equation* (9) *has no more than one integer solution if and only if b is not a divisor of a.*

Proof. It is sufficient to study the question of non-trivial integer solutions of the homogeneous equation

$$bx_{n+1} = ax_n, \quad n = 0, 1, 2, \ldots.$$

If b is a divisor of a, then this homogeneous equation has infinitely many solutions: $x_n = (\frac{a}{b})^n x_0$, where $x_0 \in \mathbb{Z}$, $n = 0, 1, 2, \ldots$.

Now suppose that b is not a divisor of a and $\{x_n\}$ is an integer solution of the homogeneous equation. Then we have:

$$b^n x_n = a^n x_0, \quad n = 0, 1, 2, \ldots.$$

Since b^n is not a divisor of a^n, there exists a prime divisor p of b such that p^n is divisor of x_0 for all $n = 0, 1, 2, \ldots$. Therefore $x_0 = 0$, and we obtain that $x_n = 0$ for all $n = 0, 1, 2, \ldots$. □

Corollary 3. *Suppose $a, b, f \in \mathbb{Z}$, $b \neq 0$ and b is not a divisor of a. The difference equation*

$$bx_{n+1} = ax_n + f, \quad n = 0, 1, 2, \ldots, \tag{10}$$

has an integer solution if and only if $b - a$ is a divisor of f. In this case the unique solution of Equation (10) *is constant:*

$$x_n = \frac{f}{b - a}, \quad n = 0, 1, 2, \ldots.$$

Proof. This assertion follows from Theorems 5 and 6. □

Theorem 7. *The Equation* (9) *has the unique solution in integer numbers for all sequences of integer numbers* $\{f_n\}$ *if and only if* $b = 0$ *and* $a = \pm 1$.

Proof. Obviously, when $b = 0$ and $a = \pm 1$, this equation has the unique solution. Let us prove the statement of the theorem in the inverse way.

Let $\{f_n\} = \{f_0, 0, 0, \dots\}$, where $f_0 \in \mathbb{Z}$. Let $\{x_n\}$ be a unique solution in integer numbers of this equation with the right-hand side $\{f_n\}$. Then sequence x_1, x_2, x_3, ... is a solution of the homogeneous equation $bx_{n+1} = ax_n$. By the uniqueness of an integer solution, we obtain that $x_n = 0$, $n = 1, 2, 3, \dots$. Thus x_0 is a solution of the equation $ax_0 = -f_0$. We obtain that this equation has a solution in integer numbers for any integer f_0. Therefore $a = \pm 1$.

Let $a = 1$ (case $a = -1$ can be considered analogously). Now suppose that $m \in \mathbb{Z}$, $m \neq 0$, $f_n = m^n f_0$, and $\{x_n\}$ is a solution in integer numbers of the equation $bx_{n+1} = x_n + f_n$. Let $y_n = x_n - mx_{n-1}$, $n = 1$, 2, 3, ... Then the sequence of integers y_1, y_2, y_3, ... is a solution of the implicit equation $by_{n+1} = y_n$, $n = 1, 2, 3, \dots$. Therefore $y_n = 0$, that is $x_n = m^n x_0$, $n = 1, 2, 3, \dots$. Hence, x_0 is an integer solution of the equation $(bm - 1)x_0 = f_0$. Since this equation has an integer solution for any integer f_0, we obtain that $bm - 1 = \pm 1$ for all $m \neq 0$. It is possible only in the case $b = 0$. □

We can consider this result as an algebraic analogue of Theorem 2 and Corollary 2 from [5].

3. The case of the second-order equation

In this section we consider a problem of a solution in integers for the following second-order implicit difference equation:

$$cx_{n+2} = bx_{n+1} + ax_n + f, \quad n = 0, 1, 2, \dots, \tag{11}$$

where $a, b, c, f \in \mathbb{Z}$. First of all we note that in the following interesting case the sufficient condition from Corollary 1 is necessary.

Proposition 1. *Let* a, b, c *be natural numbers. Then the roots of the characteristic polynomial* $c\lambda^2 - b\lambda - a$ *belong to the ring* $0 < |\lambda| < 1$ *if and only if* $c > a + b$.

Proof. The discriminant D of the equation $c\lambda^2 - b\lambda - a = 0$ is $b^2 + 4ac$, and $D > 0$ since a, c are natural. Hence the roots of this equation are real. Now elementary arguments show that the inequality $c > a + b$ follows from the condition $\lambda_1, \lambda_2 \in (-1, 1)$. □

Corollary 4. *Suppose a, b, c are natural numbers, λ_1 is a root of the characteristic polynomial of the difference equation $cx_{n+2} = bx_{n+1} + ax_n$, and $\lambda_1 \in (0, 1)$. Then this difference equation has only trivial solution.*

Proof. We have that $\lambda_1 = \frac{b+\sqrt{D}}{2c}$. Hence $\sqrt{D} < 2c - b$. Therefore

$$D < 4c^2 - 4bc + b^2, \quad \text{i.e.,} \quad c > a + b.$$

Now we obtain our statement from Proposition 1 and Theorem 1. □

Remark 1. By using the Routh–Hurwitz criterion of the stability for polynomial we can obtain the following statement.

Let a, b, c be integers. Then the roots of the characteristic polynomial $c\lambda^2 - b\lambda - a$ belong to the ring $0 < |\lambda| < 1$ if and only if $b + c - a \neq 0$, $a \neq 0$ and

$$\begin{cases} \dfrac{a+c}{c+b-a} > 0, \\ \dfrac{c-b-a}{c+b-a} > 0. \end{cases}$$

Let us consider some examples.

Example 1. Consider the following explicit difference equation

$$x_{n+2} = x_{n+1} + x_n.$$

This equation has infinitely many integer solutions. In this example the characteristic equation has the following form: $\lambda^2 - \lambda - 1 = 0$. We have $\lambda_1 = \frac{1+\sqrt{5}}{2} > 1$, $\lambda_2 = \frac{1-\sqrt{5}}{2} < 0$. If $x_0 = 0$, $x_1 = 1$, we obtain the Fibonacci sequence: $x_n = \frac{1}{\sqrt{5}}\big((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n\big)$.

Example 2. Consider now such implicit difference equation

$$2x_{n+2} = x_{n+1} + x_n, \quad n = 0, 1, 2, \ldots. \tag{12}$$

In this case $\lambda_1 = 1$, $\lambda_2 = -\frac{1}{2}$. Conditions of the theorems from Section 1 are not fulfilled, and our equation has the non-trivial solutions $x_n = m$, $n = 0, 1, 2, \ldots$, where $m \in \mathbb{Z}$.

Example 3. Let the difference equation be of the following form

$$2x_{n+2} = -x_{n+1} - x_n, \quad n = 0, 1, 2, \ldots. \tag{13}$$

The corresponding characteristic equation $2\lambda^2 + \lambda + 1 = 0$ has the roots $\lambda_{1,2} = \frac{-1 \pm i\sqrt{7}}{4}$, and these roots belong to the disk $|\lambda| < 1$. Now we can use the Theorem 1 or Corollary 2. Hence our equation has only trivial integer solution. Let us note that in this example the condition of Corollary 1 is not fulfilled.

Example 4. Consider the difference equation

$$2x_{n+2} = -5x_{n+1} - x_n, \quad n = 0, 1, 2, \dots. \tag{14}$$

The corresponding characteristic equation $2\lambda^2 + 5\lambda + 1 = 0$ has roots $\lambda_1 = \frac{-5+\sqrt{17}}{4}$, $\lambda_2 = \frac{-5-\sqrt{17}}{4}$, and $\lambda_2 < -1$. Now we cannot use Theorem 1. According to Corollary 2 our equation has only trivial integer solution. If we consider the following non-homogeneous equation

$$2x_{n+2} = -5x_{n+1} - x_n + f,$$

where $f \in \mathbb{Z}$, then this equation has an integer solution if and only if f is divisible by 8. This solution is $x_n = \frac{f}{8}$, $n = 0, 1, 2, \dots$.

Example 5. Consider the difference equation

$$6x_{n+2} = 17x_{n+1} - 12x_n, \quad n = 0, 1, 2, \dots. \tag{15}$$

The corresponding characteristic equation $6\lambda^2 - 17\lambda + 12 = 0$ has the roots $\frac{3}{2}$, $\frac{4}{3}$, which belong to the $(\frac{1}{2}, 1) \cup (1, 2)$. In this case Theorem 4 works only. Let $f \in \mathbb{Z}$. By Theorem 5, the non-homogeneous equation

$$6x_{n+2} = 17x_{n+1} - 12x_n + f$$

has the unique integer solution $x_n = f$, $n = 0, 1, 2, \dots$.

Example 6. Let now the difference equation be of the form:

$$9x_{n+2} = 15x_{n+1} - 4x_n. \tag{16}$$

Then $\lambda_1 = \frac{1}{3}$, $\lambda_2 = \frac{4}{3}$, and in this example we can use the Theorem 2 only.

4. Difference equation of the first order over the ring of p-adic integers

Suppose a and b are integers, $a \neq 0$, $b \neq 0$, $b \neq \pm 1$, b is not a divisor of a. Suppose also p is a prime, p is a divisor of b, but p is not a divisor of a. In this section we study Equation (9) over the ring of p-adic integers $\mathbb{Z}_p$ (see [3, Section 3]). We note that $\frac{1}{a} \in \mathbb{Z}_p$ since p is not a divisor of a.

Theorem 8. *Equation* (9) *has the following unique solution over* $\mathbb{Z}_p$

$$x_n = -\sum_{j=0}^{\infty} \frac{b^j}{a^{j+1}} f_{n+j}, \quad n = 0, 1, 2, \dots, \tag{17}$$

where the series on the right-hand side converges in $\mathbb{Z}_p$.

Proof. Let us consider the homogeneous equation $bx_{n+1} = ax_n$. We obtain that $x_0 = r^n x_n$, where $r = \frac{b}{a} \in \mathbb{Z}_p$. Since r is divided by p in the ring $\mathbb{Z}_p$, we have $r^n \to 0$ and $r^n x_n \to 0$ in the topology of $\mathbb{Z}_p$ (see [3, Section 3]). Hence $x_0 = 0$. Therefore $x_n = 0$ for all n.

Now consider the series on the right-hand side of (17). Since $\frac{1}{a^{j+1}} \in \mathbb{Z}_p$, we obtain $\frac{b^j}{a^{j+1}} f_{n+j} \to 0$ as $j \to \infty$ in $\mathbb{Z}_p$. Hence the series $-\sum\limits_{j=0}^{\infty} \frac{b^j}{a^{j+1}} f_{n+j}$ converges in $\mathbb{Z}_p$ (see [3, Section 3]). It is easy to see that its sum is a solution of Equation (9). □

Example 7. Consider the equation $3x_{n+1} = x_n + 1$ from Introduction. Note that $1+3+3^2+\cdots+3^N = \frac{3^{N+1}-1}{2} \to -\frac{1}{2}$ in the ring $\mathbb{Z}_3$ as $N \to \infty$, that is $-\sum\limits_{j=0}^{\infty} 3^j = \frac{1}{2}$ in $\mathbb{Z}_3$. We see that our equation has the unique solution over $\mathbb{Z}_3$: $x_n = \frac{1}{2}$, $n = 0, 1, 2, \ldots$.

References

1. V. N. Berestovskiĭ and Yu. G. Nikonorov, *Цепные дроби, группа* $\mathrm{GL}(2,\mathbb{Z})$ *и числа Пизо* [*Continued fractions, the group* $\mathrm{GL}(2,\mathbb{Z})$*, and Pisot numbers*], Мат. тр. **10** (2007), no. 1, 97–131 = Siberian Adv. Math. **17** (2007), no. 4, 268–290.
2. R. P. Boas, *Entire functions*, Acad. Press, 1954.
3. Z. I. Borevich and I. R. Shafarevich, *Number theory*, Acad. Press, 1966.
4. S. Gefter and A. Goncharuk, *Generalized backward shift operators on the ring* $\mathbb{Z}[[x]]$*, Cramer's rule for infinite linear system, and p-adic integers*, The Diversity and Beauty of Applied Operator Theory, Oper. Theory Adv. Appl., vol 268, Birkhäuser, Cham, 2018, pp. 247–259.
5. S. L. Gefter and A. L. Piven, *Неявное линейное разностное уравнение в пространствах Фреше* [*Implicit linear difference equation in Frechet spaces*], Допов. Нац. акад. наук Укр. [Dopov. Nats. Akad. Nauk Ukr.] **6** (2017), 3–8 (in Russian).
6. V. Herasimov, S. Gefter, and A. Rybalko, *Неявне лінійне неоднорідне функціональне рівняння з оператором Помм'е в кільці* $\mathbb{Z}[[x]]$ [*Implicit linear non-homogeneous functional equation with the operator of Pommies in the ring* $\mathbb{Z}[[x]]$], Буковин. мат. журн. [Bukovinian Math. J.] **4** (2016), no. 3-4, 36–39 (in Ukrainian).
7. P.-J. Cahen and J.-L. Chabert, *Integer-valued polynomials*, Math. Surveys Monogr., vol. 48, Amer. Math. Soc., Providence, RI, 1997.

Kharkiv Gymnasium No. 47, 7 Kosmonavtiv St., Kharkiv, 61000, Ukraine
Email: martsenyukvv@gmail.com

School of Mathematics and Computer Science, V. N. Karazin Kharkiv National University, 4 Svobody Sq., Kharkiv, 61022, Ukraine
Email: gefter@karazin.ua

School of Mathematics and Computer Science, V. N. Karazin Kharkiv National University, 4 Svobody Sq., Kharkiv, 61022, Ukraine
Email: aleksei.piven@karazin.ua

Voronoï's Impact on Modern Science, Book 6, Volume 1
Proceedings of the Sixth International Conference on Analytic Number Theory and Spatial Tessellations

ON RIESZ MEANS FOR THE COEFFICIENTS OF HYBRID SYMMETRIC SQUARE L-FUNCTIONS

Olga Savastru (*Ukraine*)

Abstract. Let $f(z) = \sum_{n=1}^{\infty} a_f(n)e^{2\pi inz}$ be a holomorphic cusp form of even weight $k \geq 12$ for the full modular group $SL(2,\mathbb{Z})$, $z \in H$, $H = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ is the upper half-plane. We suppose that $f(z)$ is a normalized eigenfunction for the Hecke operators $T(n)$ $(n \geq 1)$ with $a_f(1) = 1$. We study the error term in the asymptotic formula for the coefficients of hybrid symmetric square L-functions associated with f. The Voronoi-type formulas for coefficients, both of the infinite series type and of the truncated type, are proved.

Introduction

Let

$$f(z) = \sum_{n=1}^{\infty} a_f(n)e^{2\pi inz}$$

be a holomorphic cusp form of even weight $k \geq 12$ for the full modular group $SL(2,\mathbb{Z})$, $z \in H$, and $H = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ is the upper half-plane. We suppose that $f(z)$ is a normalized eigenfunction for the Hecke operators $T(n)$ $(n \geq 1)$ with $a_f(1) = 1$. In this case, $f(z)$ has the Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n)e^{2\pi inz},$$

where

$$\lambda_f(n) = \frac{a_f(n)}{n^{\frac{k-1}{2}}}, \quad T(n)f = \lambda_f(n)f$$

for every $n \in \mathbb{N}$.

The Fourier coefficients of f are known to be real.

Received August 15, 2018; revised November 20, 2018.
2010 *Mathematics Subject Classification.* Primary 11F30; Secondary 11F66.
Key words and phrases. Holomorphic cusp forms, symmetric square L-function, functional equation, Riesz means, Voronoi-type formulas.

In 1974, Deligne [1] proved

$$|\lambda_f(n)| \le \tau(n),$$

where $\tau(n)$ is the number of positive divisors of n. For prime p, we have

$$\lambda_f(p) = \alpha_p + \overline{\alpha_p}, \quad \alpha_p \cdot \overline{\alpha_p} = 1.$$

The existence of α_p is implied by Deligne's result [1].

The problem of studying the coefficients $\lambda_f(n)$ on special sequences was considered by various authors. Hafner and Ivić [4] obtained the O-estimate and $\Omega_\pm$-results for

$$\sum_n \lambda_f(n).$$

Rankin [11, 12], Selberg [13] investigated the second moment

$$\sum_{n\le x} |\lambda_f(n)|^2.$$

Rankin considered also higher moments

$$\sum_{n\le x} \lambda_f(n)^{2\beta},$$

where $0 < \beta < 1$ and $\beta > 1$.

In 2011, Lau, Lü, and Wu [10] obtained estimates for

$$\sum_{n\le x} \lambda_f(n)^j,$$

where $3 \le j \le 8$.

Results for the sums

$$\sum_{n\le x} \lambda_f(n^2) \quad \text{and} \quad \sum_{n\le x} \lambda_f^2(n^j),$$

where $j = 2, 3, 4$, can be found in [8, 9].

In [14], Shimura introduced the function $L(s, \operatorname{sym}^2 f, \chi)$. For an arbitrary primitive Dirichlet character χ of modulus d, the symmetric square L-function attached to f is defined as the following Euler product:

$$L(s, \operatorname{sym}^2 f, \chi) := \prod_p (1 - \alpha_f^2(p)\chi(p)p^{-s})(1 - \chi(p)p^{-s})^{-1} \times (1 - \overline{\alpha}_f^2(p)\chi(p)p^{-s})$$

for $\Re s > 1$.

We have

$$L(s, \operatorname{sym}^2 f, \chi) = L(2s, \chi^2) \sum_{n=1}^{\infty} \frac{\lambda_f(n^2)\chi(n)}{n^s} := \sum_{n=1}^{\infty} \frac{c_n \chi(n)}{n^s},$$

where $L(2s, \chi^2)$ is the Dirichlet L-function associated with χ^2.

We will investigate the distribution of coefficients of a hybrid symmetric square L-function, consider the Riesz means for the coefficients of $L(s, \operatorname{sym}^2 f, \chi)$. The Voronoi-type formulas, both of the infinite series type and of the truncated type, will be proved. Voronoi-type formulas allowing one to consider the exponential sums connected with $c_n\chi(n)$ can be used for the evaluation of a corresponding sum over arithmetic progressions.

Fomenko [2] investigated the sum of coefficients of a symmetric square L-function associated with a cusp form and a trivial character. Wang [16] studied the Riesz means of the coefficients of an L-function with the mth symmetric power. Ichihara [5] considered the sum of coefficients of twisted Rankin–Selberg L-series.

1. Auxiliary statements

Now, we introduce some facts about the hybrid symmetric square L-function.

Let χ be a primitive Dirichlet character χ mod d. It is well known that the L-function $L(s, \operatorname{sym}^2 f, \chi)$ extends to an entire function and satisfies the following functional equation [14]:

$$\Lambda(s, \operatorname{sym}^2 f, \chi) = C_\chi \Lambda(1 - s, \operatorname{sym}^2 f, \bar{\chi}), \tag{1}$$

where

$$\Lambda(s, \operatorname{sym}^2 f, \chi) = \left(\frac{\pi}{d}\right)^{-\frac{3s}{2}} \Gamma\left(\frac{s}{2} + \frac{1}{2}\right) \times \Gamma\left(\frac{s}{2} + \frac{k-1}{2}\right) \Gamma\left(\frac{s}{2} + \frac{k}{2}\right) L(s, \operatorname{sym}^2 f, \chi)$$

and C_χ is a constant depending on χ with $|C_\chi| = 1$. Particularly, when χ is a primitive character with odd prime modulus p, $C_\chi = \frac{W(\chi)^3}{p^{\frac{3}{2}}}$ with the Gauss sum $W(\chi)$.

Lemma 1. *Let p be an odd prime, $b \in \mathbb{N}$, $(b,p)=1$. Then we have*

$$\sum_{\chi \bmod p} C_\chi \chi(b) \ll \varphi(p) p^{-\frac{1}{2}} = \frac{W(\chi)^3}{p^{\frac{3}{2}}},$$

where the sum is running over primitive characters $\bmod p$.

Proof.

$$\sum_{\chi \bmod p} C_\chi \chi(b) = \frac{1}{p^{\frac{3}{2}}} \sum_{\chi \bmod p} W(\chi)^3 \chi(b)$$

$$= \frac{1}{p^{\frac{3}{2}}} \sum_{\chi \bmod p} \sum_{y_1,y_2,y_3=1}^{p-1} \chi(y_1y_2y_3b) \times \exp\left(\frac{2\pi i(y_1+y_2+y_3)}{p}\right) + O(1)$$

$$= \frac{\varphi(p)}{p^{\frac{3}{2}}} \sum_{\substack{y_1,y_2,y_3=1 \\ y_1y_2y_3 \equiv b^{-1}(p)}}^{p-1} \exp\left(\frac{2\pi i(y_1+y_2+y_3)}{p}\right) + O(1).$$

To obtain the assertion, we apply the Deligne's estimation

$$\left| \sum_{\substack{y_1,y_2,y_3=1 \\ y_1y_2y_3 \equiv b^{-1}(p)}}^{p-1} \exp\left(\frac{2\pi i(y_1+y_2+y_3)}{p}\right) \right| \leq 3 \cdot p. \qquad \square$$

Throughout this paper, ϵ is an arbitrarily small positive constant.

Lemma 2. *The estimate*

$$L(\sigma+it, \operatorname{sym}^2 f, \chi) \ll (d(|t|+1))^{\frac{3}{2}(1-\sigma+\epsilon)} \tag{2}$$

holds uniformly for $-\epsilon \leq \sigma \leq 1+\epsilon$.

Proof. It is known that $|c_n| \leq d_3(n)$, where $d_3(n)$ is the number of ways to write n as a product of 3 factors. Then we get

$$L(1+\epsilon+it, \operatorname{sym}^2 f, \chi) \ll \log(|t|+1)^3,$$

From the functional equation (1) for $s = -\epsilon + it$, we have

$$L(s, \operatorname{sym}^2 f, \chi) \ll \left| d^{\frac{3}{2}(1-2s)} \frac{\Gamma\left(\frac{1-s}{2}+\frac{1}{2}\right)\Gamma\left(\frac{1-s}{2}+\frac{k-1}{2}\right)\Gamma\left(\frac{1-s}{2}+\frac{k}{2}\right)}{\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)\Gamma\left(\frac{s}{2}+\frac{k-1}{2}\right)\Gamma\left(\frac{s}{2}+\frac{k}{2}\right)} \right| \times \left| L(1+\epsilon+it, \operatorname{sym}^2 f, \chi) \right|$$

$$\ll (d(|t|+1))^{\frac{3}{2}(1+2\epsilon)} \log(|t|+1)^3.$$

Applying the Phragmén–Lindelöf principle, we obtain the statement of Lemma 2. $\square$

2. Voronoi-type formulas

When χ is a primitive character, we define the Riesz mean of the coefficients of the hybrid symmetric square L-function as follows:

$$D_\rho(x, \mathrm{sym}^2 f, \chi) = \Gamma(\rho+1)^{-1} \sum_{n \le x}{}' c_n \chi(n)(x-n)^\rho,$$

where $x \ge 1$ and $\rho \ge 0$ is a real number. $\sum'$ means that if $\rho = 0$ and x is an integer, then c_x in the sum is replaced by $\frac{1}{2}c_x$.

Hafner [3] obtained the Voronoi-type formulas for the Riesz mean of general L-functions with functional equations. The following Voronoi formula is obtained by applying his result to $D_\varrho(x, \mathrm{sym}^2 f, \chi)$. In accordance with [3], we represent the Riesz mean as follows:

$$D_\rho(x, \mathrm{sym}^2 f, \chi) = \frac{L_\rho(0, \mathrm{sym}^2 f, \chi)}{\Gamma(\rho+1)} x^\rho + \Delta_\rho(x, \mathrm{sym}^2 f, \chi). \tag{3}$$

For $\rho = 0$, we have

$$\sum_{n \le x}{}' c_n \chi(n)) = L(0, \mathrm{sym}^2 f, \chi) + \Delta_0(x, \mathrm{sym}^2 f, \chi).$$

We can apply Theorems A and B of [3] to the function $L(s, \mathrm{sym}^2 f, \chi)$ with (in Hafner's notation)

$$a(n) = b(n) = c_n \chi(n),\ \lambda_n = \mu_n = (\frac{\pi}{d})^{\frac{3}{2}} n,$$

$$\varphi(s) = \psi(s) = (\frac{\pi}{d})^{-\frac{3}{2}s} L(s, \mathrm{sym}^2 f, \chi),$$

$$\Delta(s) = \Gamma\left(\frac{s}{2} + \frac{1}{2}\right) \Gamma\left(\frac{s}{2} + \frac{k-1}{2}\right) \Gamma\left(\frac{s}{2} + \frac{k}{2}\right),$$

$$\sigma_a = \sigma_a^\star = 1,\ N = 3,\ \alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{2},$$

$$\alpha = \frac{3}{2},\ r = 1,\ \beta_1 = \frac{1}{2}, \beta_2 = \frac{k-1}{2}, \beta_3 = \frac{k}{2},$$

$$a = -\frac{1}{3},\ h = 6,\ \theta_\rho = \frac{2}{3}\rho + \frac{1}{3}.$$

Let L be the oriented polygonal path with vertices $a - i\infty$, $a - iT$, $b - iT$, $b + iT$, $a + iT$, and $a + i\infty$ (in that order), where b and T are real numbers satisfying $b > k$, $T > k$, and b is not an integer.

We define the integral

$$f_\rho(y) = \frac{1}{2\pi i} \int_L \frac{\Gamma(1-s)\Delta(s)}{\Gamma(2+\rho-s)\Delta(1-s)} y^{1+\rho-s} ds$$

$$= \frac{1}{2\pi i}\int\limits_{L} \frac{\Gamma(1-s)\Gamma(\frac{s+1}{2})\Gamma(\frac{s+k-1}{2})\Gamma(\frac{s+k}{2})}{\Gamma(2+\rho-s)\Gamma(\frac{-s}{2})\Gamma(\frac{-s+k+1}{2})\Gamma(\frac{-s+k}{2})} y^{1+\rho-s} ds$$

for $y > 0$.

The following result is due to Hafner [3]:

$$\Delta_\rho(x, \mathrm{sym}^2 f, \chi) = \left(\frac{\pi}{d}\right)^{\frac{3}{2}(2\rho+1)} \sum_{n=1}^{\infty} \frac{C_\chi c_n \bar{\chi}(n)}{n^{1+\rho}} f_\rho\left(\left(\frac{\pi}{d}\right)^3 nx\right)$$

for $x > 0$, where the infinite series on the right-hand side converges for $\rho > 0$ uniformly on any finite closed x-interval.

Using Lemmas 2.1 and 2.2 [3], we have

$$\frac{d}{dx} f_{\rho+1}(x) = f_{\rho+1}(x)$$

and

$$f_\rho(y) = 2^{-\rho}(3\pi)^{-\frac{1}{2}} y^{\frac{2}{3}\rho+\frac{1}{3}} cos(6y^{\frac{1}{3}} - \frac{\pi}{2}\rho) + O(y^{\frac{2}{3}\rho}) + O(y^{1+\rho-b}),$$

where $b > k$, $\rho \geq 0$, and the last term does not appear, when $b > \rho + 1$ and ρ is not an integer.

We note that the correct value $e_0(\rho)$ in Lemma 2.1 [3] should be

$$e_0(\rho) = \left(\frac{2\alpha}{h}\right)^\rho \left(\frac{2}{6h}\right)^{\frac{1}{2}} = 2^{-\rho}(3\pi)^{-\frac{1}{2}}.$$

Now, we present the truncated Voronoi formula for $\Delta_\rho(x, \mathrm{sym}^2 f, \chi)$.

Theorem 1. *Let $x > 1$, $0 \leq \rho \leq 1$, $N \geq d^3$, and let N be large enough compared with k and ρ. Then we have the following truncated Voronoi formula:*

$$\begin{aligned}
&\Delta_\rho(x, \mathrm{sym}^2 f, \chi) \\
&\quad = C_\chi d^{\rho+\frac{1}{2}} 2^{-\rho} \pi^{-\rho-1} 3^{-\frac{1}{2}} x^{\frac{2}{3}\rho+\frac{1}{3}} \sum_{n \leq N} \frac{c_n \bar{\chi}(n)}{n^{\frac{1}{3}\rho+\frac{2}{3}}} \cos(6\pi d^{-1}(nx)^{\frac{1}{3}} - \frac{\pi}{2}\rho) \\
&\qquad + O(x^{\frac{4\rho+1}{6}} d^{\rho+1} N^{\frac{1-2\rho}{6}+\epsilon}) \\
&\qquad + O(x^{\frac{2(1+\rho)}{3}+\epsilon} d^{1+\rho+\epsilon} N^{-\frac{1+\rho}{3}+\epsilon}) \\
&\qquad + O(x^{\frac{2\rho}{3}} d^{\frac{3}{2}+\rho} N^{-\frac{\rho}{3}}) + O(x^{\rho+\epsilon}) + O(x^\rho d^{\frac{3}{2}}).
\end{aligned}$$

The last error term does not appear in the case

$$\frac{d^3}{\pi^3 x} \leq 1.$$

Proof. We put $c = 1 + \epsilon$ and $T > 2 + k$. We have

$$D_\rho(x, \mathrm{sym}^2 f, \chi) = I_\rho(x, c) + O(x^{\rho+\epsilon}) + O(T^{-1-\rho}x^{\rho+1+\epsilon}), \tag{4}$$

where

$$I_\rho(x, c) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{\Gamma(s)}{\Gamma(s+\rho+1)} L(s, \mathrm{sym}^2 f, \chi) x^{s+\rho} ds.$$

We replace the path of $I_\rho(x, c)$ by the rectangle joining the points $c - iT$, $-\epsilon - iT$, $-\epsilon + iT$, and $c + iT$. The integrals over the horizontal sides of the rectangle can be estimated, by using Lemma 2:

$$\int_{-\epsilon-iT}^{1+\epsilon+iT} \frac{\Gamma(s)}{\Gamma(1+\rho+s)} L(s, \mathrm{sym}^2 f, \chi) x^{s+\rho} ds \ll x^{1+\epsilon+\rho} T^{-1-\rho+3\epsilon} d^{3\epsilon}$$

$$+ x^{\rho-\epsilon} T^{\frac{1}{2}-\rho+3\epsilon} d^{\frac{3}{2}+3\epsilon}.$$

By using the residue theorem, we obtain

$$I_\rho(x, c) = \frac{x^\rho}{\Gamma(\rho+1)} L(0, \mathrm{sym}^2 f, \chi) + I_\rho(x, -\epsilon)$$

$$+ O(T^{-1-\rho}x^{\rho+1+\epsilon}d^{3\epsilon}) + O(T^{\frac{1}{2}-\rho+3\epsilon}x^{\rho-\epsilon}d^{\frac{3}{2}+3\epsilon}). \tag{5}$$

Applying the functional equation (3), we find

$$I_\rho(x, -\epsilon) = \frac{C_\chi}{2\pi i} \left(\frac{\pi}{d}\right)^{\frac{3}{2}} x^{1+\rho} \sum_{n=1}^{\infty} c_n \bar{\chi}(n) K_\rho(n) \tag{6}$$

with

$$K_\rho(n) = \frac{1}{2\pi i} \int_{-\epsilon-iT}^{-\epsilon+iT} \frac{\Gamma(s)\Gamma(\frac{-s}{2})\Gamma(\frac{-s+k}{2})\Gamma(\frac{1-s+k}{2})}{\Gamma(1+\rho+s)\Gamma(\frac{1+s}{2})\Gamma(\frac{s+k-1}{2})\Gamma(\frac{s+k}{2})} \left(\frac{\pi^3 xn}{d^3}\right)^{s-1} ds.$$

We divide the sum on the right-hand side of (6) into $n \leq N$ and $n > N$. Then we choose $T = \frac{\pi}{d}(x(N + \frac{1}{2}))^{\frac{1}{3}}$. Using Stirling's and the first derivative test (Lemma 2.1 of [6]), we have

$$F(t) = -3t(\ln t - \ln 2 - 1) + t \ln(nx).$$

We can find that $F'(t) \geq \log \frac{n}{N+\frac{1}{2}}$. Hence, the integral is

$$K_\rho(n) \ll \left(\frac{d^3}{nx}\right)^{1+\epsilon} T^{\frac{1}{2}+3\epsilon-\rho} \left(1 + \left(\log \frac{n}{N+\frac{1}{2}}\right)\right)$$

for $n > N$. This estimate yields

$$I_\rho(x,-\epsilon) = \frac{C_\chi}{2\pi i}\left(\frac{\pi}{d}\right)^{\frac{3}{2}} x^{1+\rho}\sum_{n\le N} c_n\bar{\chi}(n)K_\rho(n) + O(x^{\frac{1+4\rho}{6}}d^{1+\rho}N^{\frac{1-2\rho}{6}+\epsilon}). \quad (7)$$

We divide the sum on the right-hand side of (7) into two parts $\frac{d^3}{\pi^3 nx} \le n \le N$ and $n < \frac{d^3}{\pi^3 x}$ in the case $\frac{d^3}{\pi^3 x} > 1$. In the case $\frac{\pi^3 nx}{d^3} \ge 1$, we can use the fact that $K_\rho(n)$ can be expressed by $f_\rho(\frac{\pi^3 nx}{d^3})$. First, we suppose $\frac{d^3}{\pi^3 x} > 1$. We change the path of integration of $K_\rho(n)$ to the path L, which is an oriented polygon path with vertices $\lambda - i\infty$, $\lambda - iT$, $-\epsilon - iT$, $-\epsilon + iT$, $\lambda + iT$, $\lambda + i\infty$, where $\frac{1}{6} < \lambda < 1$ (this bound of λ depends on the use of the first derivative test). The resulting integral is equal to $(\pi^3 d^{-3}nx)^{-\rho-1}f_\rho(\frac{\pi^3 nx}{d^3})$. Then we have

$$\begin{aligned} I_\rho(x,-\epsilon) &= \frac{C_\chi}{2\pi i}\left(\frac{\pi}{d}\right)^{\frac{3}{2}} x^{1+\rho}\sum_{\frac{d^3}{\pi^3 nx}\le n\le N} c_n\bar{\chi}(n)(\pi^3 d^{-3}nx)^{-\rho-1}f_\rho(\frac{\pi^3 nx}{d^3}) \\ &\quad + \frac{C_\chi}{2\pi i}\left(\frac{\pi}{d}\right)^{\frac{3}{2}} x^{1+\rho}\sum_{n<\frac{d^3}{\pi^3 x}} c_n\bar{\chi}(n)K_\rho(n) + O(x^{\frac{1+4\rho}{6}}d^{1+\rho}N^{\frac{1-2\rho}{6}+\epsilon}) \\ &= C_\chi d^{\rho+\frac{1}{2}}2^{-\rho}\pi^{-\rho-1}3^{-\frac{1}{2}}x^{\frac{2}{3}\rho+\frac{1}{3}}\sum_{\frac{d^3}{\pi^3 nx}\le n\le N}\frac{c_n\bar{\chi}(n)}{n^{\frac{1}{3}\rho+\frac{2}{3}}}\cos(6\pi d^{-1}(nx)^{\frac{1}{3}} - \frac{\pi}{2}\rho) \\ &\quad + \frac{C_\chi}{2\pi i}\left(\frac{\pi}{d}\right)^{\frac{3}{2}} x^{1+\rho}\sum_{n<\frac{d^3}{\pi^3 x}} c_n\bar{\chi}(n)K_\rho(n) + O(x^{\frac{1+4\rho}{6}}d^{1+\rho}N^{\frac{1-2\rho}{6}+\epsilon}) \\ &\quad + O(x^{\frac{2\rho}{3}}d^{\frac{3}{2}+\rho}N^{-\frac{\rho}{3}}) \\ &= C_\chi d^{\rho+\frac{1}{2}}2^{-\rho}\pi^{-\rho-1}3^{-\frac{1}{2}}x^{\frac{2}{3}\rho+\frac{1}{3}}\sum_{\frac{d^3}{\pi^3 nx}\le n\le N}\frac{c_n\bar{\chi}(n)}{n^{\frac{1}{3}\rho+\frac{2}{3}}}\cos(6\pi d^{-1}(nx)^{\frac{1}{3}} - \frac{\pi}{2}\rho) \\ &\quad + O(x^\rho d^{\frac{3}{2}}) + O(x^{\frac{1+4\rho}{6}}d^{1+\rho}N^{\frac{1-2\rho}{6}+\epsilon}) + O(x^{\frac{2\rho}{3}}d^{\frac{3}{2}+\rho}N^{-\frac{\rho}{3}}). \end{aligned} \quad (8)$$

Second, we suppose $\frac{d^3}{\pi^3 x} \le 1$. We obtain (8) in the same way, but the sum for $n < \frac{d^3}{\pi^3 x}$ does not appear. From (4), (5), (7), and (8) we get the assertion of Theorem 1. □

Theorem 2. *Let $X > 1$ be a real number. For any fixed ρ, $0 \le \rho \le 1$, we have*

$$\int_0^X |\Delta_\rho(t, \operatorname{sym}^2 f, \chi)|^2 dt = CX^{\frac{4}{3}\rho+\frac{5}{3}} + O(X^{\rho+\frac{5}{3}+\epsilon} d^{2+\rho+\epsilon} + X^{\frac{2}{3}\rho+\frac{5}{3}+\epsilon} d^{3+\epsilon}),$$

$$C = d^{2\rho+1} 2^{-2\rho-1} \pi^{-2\rho-2} (4\rho+5)^{-1} \sum_{n=1}^{\infty} \frac{|c_n|^2}{n^{\frac{2}{3}\rho+\frac{4}{3}}}.$$

Proof. Using Theorem 1, we get

$$\Delta_\rho(x, \operatorname{sym}^2 f, \chi) = \delta_N(x) + E_N(x),$$

where

$$\delta_N(x) = C_\chi d^{\rho+\frac{1}{2}} 2^{-\rho} \pi^{-\rho-1} 3^{-\frac{1}{2}} x^{\frac{2}{3}\rho+\frac{1}{3}} \sum_{n \le N} \frac{c_n \overline{\chi}(n)}{n^{\frac{1}{3}\rho+\frac{2}{3}}} \cos(6\pi d^{-1}(nx)^{\frac{1}{3}} - \frac{\pi}{2}\rho),$$

$$E_N(x) = O(x^{\frac{4\rho+1}{6}} d^{\rho+1} N^{\frac{1-2\rho}{6}+\epsilon}) + O(x^{\frac{2(1+\rho)}{3}+\epsilon} d^{1+\rho+\epsilon} N^{-\frac{1+\rho}{3}+\epsilon})$$
$$+ O(x^{\frac{2\rho}{3}} d^{\frac{3}{2}+\rho} N^{-\frac{\rho}{3}}) + O(x^{\rho+\epsilon}) + O(x^\rho d^{\frac{3}{2}}).$$

We can write

$$\int_0^X |\Delta_\rho(x, \operatorname{sym}^2 f, \chi)|^2 dx = \sum_{j=0}^{\infty} \int_{2^{-j}X}^{2^{-j+1}X} |\Delta_\rho(x, \operatorname{sym}^2 f, \chi)|^2 dx. \tag{9}$$

By the Cauchy–Schwarz inequality, we have

$$\int_Y^{2Y} |\Delta_\rho(x, \operatorname{sym}^2 f, \chi)|^2 dx \ll \int_Y^{2Y} |\delta_N(x)|^2 dx + \int_Y^{2Y} |E_N(x)|^2 dx$$
$$+ \left(\int_Y^{2Y} |\delta_N(x)|^2 dx \right)^{\frac{1}{2}} \left(\int_Y^{2Y} |E_N(x)|^2 dx \right)^{\frac{1}{2}}. \tag{10}$$

Let $N = Axd^3$, where A is a positive constant independent of x and d.

For $E_N(x)$, we have

$$E_N(x) = O(x^{\frac{1+\rho+\epsilon}{3}} d^{\frac{3}{2}+\epsilon})$$

and

$$\int_Y^{2Y} |E_N(x)|^2 dx \ll d^{3+\epsilon} Y^{\frac{2}{3}\rho+\frac{5}{3}+\epsilon}, \tag{11}$$

$$\int_Y^{2Y} |\delta_N(x)|^2 dx = d^{2\rho+1} 2^{-2\rho} \pi^{-2\rho-2} 3^{-1} \int_Y^{2Y} x^{\frac{4}{3}\rho+\frac{1}{3}} \sum_{m,n \le N} \frac{|c_m c_n|}{(mn)^{\frac{1}{3}\rho+\frac{2}{3}}}$$
$$\times \cos(6\pi d^{-1}(mx)^{\frac{1}{3}} - \frac{\pi}{2}\rho) \cos(6\pi d^{-1}(nx)^{\frac{1}{3}} - \frac{\pi}{2}\rho) dx.$$

For $m = n$, we have

$$d^{2\rho+1}2^{-2\rho}\pi^{-2\rho-2}3^{-1}\sum_{n\le N}\int_Y^{2Y} x^{\frac{4}{3}\rho+\frac{2}{3}}\frac{|c_n|^2}{n^{\frac{2}{3}\rho+\frac{4}{3}}}\cos^2(6\pi d^{-1}(nx)^{\frac{1}{3}}-\frac{\pi}{2}\rho)dx$$

$$= d^{2\rho+1}2^{-2\rho-1}\pi^{-2\rho-2}3^{-1}\sum_{n\le N}\frac{|c_n|^2}{n^{\frac{2}{3}\rho+\frac{4}{3}}}$$

$$\times\int_Y^{2Y} x^{\frac{4}{3}\rho+\frac{2}{3}}(1+\cos(12\pi d^{-1}(nx)^{\frac{1}{3}}-\pi\rho))dx$$

$$= d^{2\rho+1}2^{-2\rho-1}\pi^{-2\rho-2}(4\rho+5)^{-1}((2Y)^{\frac{4}{3}\rho+\frac{5}{3}}-(Y)^{\frac{4}{3}\rho+\frac{5}{3}})\sum_{n=1}^{\infty}\frac{|c_n|^2}{n^{\frac{2}{3}\rho+\frac{4}{3}}}$$

$$+O(d^{2\rho+1}Y^{\frac{2}{3}\rho+\frac{4}{3}+\epsilon})+O(d^{2\rho+1}Y^{\frac{4}{3}\rho+\frac{4}{3}}). \tag{12}$$

For $m \ne n$, we have

$$\sum_{m\ne n\le N}\frac{|c_mc_n|}{(mn)^{\frac{1}{3}\rho+\frac{2}{3}}}\int_Y^{2Y} x^{\frac{4}{3}\rho+\frac{2}{3}}\cos(6\pi d^{-1}(mx)^{\frac{1}{3}}-\frac{\pi}{2}\rho)$$

$$\times\cos(6\pi d^{-1}(nx)^{\frac{1}{3}}-\frac{\pi}{2}\rho)dx = \sum_{m\ne n\le N}\frac{|c_mc_n|}{(mn)^{\frac{1}{3}\rho+\frac{2}{3}}}\int_Y^{2Y} x^{\frac{4}{3}\rho+\frac{2}{3}}$$

$$\times\cos(6\pi d^{-1}(mx)^{\frac{1}{3}}-6\pi d^{-1}(nx)^{\frac{1}{3}})dx$$

$$+\sum_{m\ne n\le N}\frac{|c_mc_n|}{(mn)^{\frac{1}{3}\rho+\frac{2}{3}}}\int_Y^{2Y} x^{\frac{4}{3}\rho+\frac{2}{3}} \tag{13}$$

$$\times\cos(6\pi d^{-1}(mx)^{\frac{1}{3}}+6\pi d^{-1}(nx)^{\frac{1}{3}}-\pi\rho)dx \ll Y^{\rho+\frac{5}{3}+\epsilon}+Y^{\frac{2}{3}\rho+\frac{5}{3}+\epsilon}.$$

Combining (10)–(13), we obtain

$$\int_Y^{2Y}|\Delta_\rho(x,\mathrm{sym}^2 f,\chi)|^2dx \tag{14}$$

$$= d^{2\rho+1}2^{-2\rho-1}\pi^{-2\rho-2}(4\rho+5)^{-1}\sum_{n=1}^{\infty}\frac{|c_n|^2}{n^{\frac{2}{3}\rho+\frac{4}{3}}}((2Y)^{\frac{4}{3}\rho+\frac{5}{3}}-Y^{\frac{4}{3}\rho+\frac{5}{3}})$$

$$+O(Y^{\rho+\frac{5}{3}+\epsilon}d^{2+\rho+\epsilon}+Y^{\frac{2}{3}\rho+\frac{5}{3}+\epsilon}d^{3+\epsilon}).$$

This result and (9) yield Theorem 2. □

References

1. P. Deligne, *La conjecture de Weil.* I, Publ. Math. Inst. Hautes Études Sci. **43** (1974), 273–307.
2. O. M. Fomenko, *The behavior of Riesz means of the coefficients of a symmetric square L-function*, J. Math. Sci. **143** (2007), no. 3, 3174–3181.
3. J. L. Hafner, *On the representation of the summatory functions of a class of arithmetical functions*, Analytic Number Theory (Philadelphia, Pa., 1980), Lect. Notes Math., vol. 899, Springer, Berlin–New York, 1981, pp. 148–165.
4. J. Hafner and A. Ivić, *On sums of Fourier coefficients of cusp forms*, Enseign. Math. **35** (1989), no. 3–4, 375–382.
5. Y. Ichihara, *On the Riesz means for the coefficients of twisted Rankin–Selberg L-functions*, J. Math. Soc. Japan **55** (2003), no. 1, 81–100.
6. A. Ivić, *The Riemann zeta-function*, Wiley, New York, 1985.
7. A. Ivić, K. Matsumoto, and Y. Tanigawa, *On Riesz means of the coefficients of the Rankin–Selberg series*, Math. Proc. Cambridge Philos. Soc. **127** (1999), 117–131.
8. H. X. Lao and A. Sankaranarayanan, *The average behavior of Fourier coefficients of cusp forms over sparse sequences*, Proc. Amer. Math. Soc. **137** (2009), no. 8, 2557–2565.
9. H. X. Lao and H. Wei, *Ω-result on coefficients of automorphic L-functions over sparse sequences*, J. Korean Math. Soc. **52** (2015), no. 5, 945–954.
10. Y. K. Lau, G. S. Lü, and J. Wu, *Integral power sums of Hecke eigenvalues*, Acta Arith. **150** (2011), no. 2, 193–207.
11. R. A. Rankin, *An Ω-result for the coefficients of cusp forms*, Math. Ann. **203** (1973), 239–250.
12. R. A. Rankin, *Sums of powers of cusp form coefficients.* II, Math. Ann. **272** (1985), 593–600.
13. A. Selberg, *Bemerkungen über eine Dirichletsche Reihe, die mit der Theorie der Modulformen nahe verbunden ist*, Arch. Math. Naturvid. **43** (1940), 47–50.
14. G. Shimura, *On the holomorphy of certain Dirichlet series*, Proc. London Math. Soc. **31** (1975), 79–98.
15. E. C. Titchmarsh, *The theory of the Riemann zeta-function*, Oxford Univ. Press, Oxford, 1986.
16. H. Wang, *On the Riesz means of coefficients of mth symmetric power L-functions*, Lith. Math. J. **50** (2010), no. 4, 474–488.

Department of Computer Algebra and Discrete Mathematics, Odessa I. I. Mechnikov National University, 2 Dvorianska St., Odessa, 65026, Ukraine
Email: savolga777@gmail.com

Voronoï's Impact on Modern Science, Book 6, Volume 1
Proceedings of the Sixth International Conference on Analytic Number Theory and Spatial Tessellations

THE PARTIAL SUM OF SOME FUNCTIONS UNDER A CONSTRAINT ON THE SUM OF DIGITS OF SQUARES

Teerapat Srichan (*Thailand*)

Abstract. We extend the paper of K. Aloui to the summatory functions of some arithmetical functions under a constraint on the sum of digits of squares by elementary method.

1. Statement of the main results

In this paper $\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, non-negative integers, integers, real numbers and complex numbers respectively. If m and n are integers, then (m, n) denotes the greatest common divisor. q is, unless otherwise stated, an integer ≥ 2 and p a prime number. For a function f, we write

$$f(x) = O(g(x)) \text{ or } f(x) \ll g(x),$$

if there exists a constant $C > 0$, such that $|f(x)| \leq C|g(x)|$. We write $f(x) = O_q(g(x))$, where the implicit constant C depends on q. If x is a real number, we have $\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}$ and $\|x\| = \min\{|x-n| : n \in \mathbb{Z}\}$. We use the abbreviation $e(x) = e^{2\pi i x}$. For $s \in \mathbb{C}$, $\zeta(s)$ denote the Riemann zeta-function.

Further, $\mu(n)$ is the Möbius function,

$$\mu(n) : n \mapsto \begin{cases} 1, & \text{if } n = 1, \\ (-1)^r, & \text{if } n = p_1 p_2 \cdots p_r, \quad \text{a product of distinct primes,} \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, $\omega(n)$ is the number of distinct prime factors of a positive integer n, $\tau(n)$ is the number of positive divisors function, $\tau : n \mapsto \sum_{d|n} 1$. For fixed integers $1 \leq a \leq b \leq c$, $d(a, b, c; n)$ is the number of representations of n as $n = n_1^a n_2^b n_3^c$, where n_1, n_2, n_3 are natural numbers.

Received July 31, 2018; revised November 19, 2018.
2010 *Mathematics Subject Classification.* 11N37, 11N60, 11N69.
Key words and phrases. Sum of digits of function.

Every non-negative integer can be written uniquely in base q as

$$n = \sum_{k=0}^{\infty} a_k(n) q^k,$$

where the sequence $(a_j(n))_{j\in\mathbb{N}_0} \in \{0, 1, ..., q-1\}^{\mathbb{N}_0}$. The sum of digits function in base q is defined by

$$S(n) = S_q(n) = \sum_{k=0}^{\infty} a_k(n).$$

The sum of digits function S is an interesting example of q-additive functions, namely $S(aq^j+b) = S(a)+S(b)$, with $(a, b, j) \in \mathbb{N}^3, 0 \leq b \leq q^j$. Such functions were introduced by Gelfond [3] and further studied by many mathematicians. We refer the reader to the monograph [8] (see Chapter IV, part 4.3) for more details.

Gelfond [3] studied the distribution of the sum of digits function in residue classes and showed that, for $q, m \geq 2$ and a, l, r integers such that $(m, q-1) = 1$, we have

$$\#\{1 \leq n \leq N : n \equiv l \bmod r, S(n) \equiv a \bmod m\} = \frac{N}{mr} + O_q(N^\lambda),$$

where $\lambda = \frac{1}{2\log q} \log \frac{q\sin(\pi/2m)}{\sin(\pi/2mq)} < 1$ is a positive constant depending only on q and m. In his paper, Gelfond remarked that it would be interesting to know the distribution of the sum of digits function of primes. In 2010, Mauduit and Rivat [6] showed that

$$\#\{p \leq x : p \text{ prime and}, S(p) \equiv a \bmod m\} = \frac{d}{m}\pi(x; d, a) + O_{q,m}(x^{1-\sigma_{q,m}}),$$

where $\pi(x; d, a) = \#\{p \leq x | p \equiv a \bmod d\}$, $d = (q-1, m)$ and $\sigma_{q,m}$ is a positive constant.

Moreover, Gelfond stated the interesting problem of finding an estimate of the number of values of a polynomial P satisfying the condition $S(P(n)) \equiv a \bmod m$. In 2009, Mauduit and Rivat [5] solved Gelfond's problem in the case $P(n) = n^2$. They proved that

$$\#\{n \leq x : S(n^2) \equiv a \bmod m\} = \frac{x}{m} Q(a, d) + O_{q,m}(x^{1-\sigma_{q,m}}),$$

where $d = (q-1, m)$, $Q(a, d) = \#\{0 \leq n \leq d | n^2 \equiv a \bmod d\}$ and $\sigma_{q,m}$ is a positive constant. In 2014 Mkaouar and Wannes [7] studied the average of some additive functions under digital constraints. Namely,

they showed that, for every $a \in \mathbb{Z}$, $x \geq 2$, and $(b, q-1) = 1$,

$$\sum_{n \leq x} \sum_{\substack{p \mid n \\ S(p) \equiv a \bmod b}} 1 = \frac{x}{b} \log\log x + \beta x + O\left(\frac{x}{\log x}\right),$$

where

$$\beta = \sum_{j=1}^{b-1} e\left(\frac{-jr}{b}\right) \int_2^{\infty} \left(\sum_{p \leq t} e\left(\frac{jS(p)}{b}\right)\right) \frac{dt}{t^2}.$$

They used a result of Drmota, Mauduit-Rivat [2] and Abel's summation formula. Moreover, the average of the total number of prime factors of a positive integer under digital constraints is given in their paper. In 2016 K. Aloui [1] use a similar approach in [7] to study the average of some multiplicative functions (namely, the number of positive divisors function, the sum of the s-th powers of all positive divisors function, the number of positive integer $\leq n$ and coprime to n and the number of representations of n as the sum of two integral squares) under constraints on the sum of the digits of squares. To this, Aloui used the following result of Mauduit and Rivat [5] and Abel's summation formula.

Theorem 1 (Mauduit-Rivat [5]). *Let $q \geq 2$ be an integer and $\alpha \in \mathbb{R}$ such that $(q-1)\alpha \in \mathbb{R}/\mathbb{Z}$ then there exists $\sigma_q(\alpha) > 0$ and $x_0 := x_0(q, \alpha) \geq 2$ such that for every real number $x \geq x_0$, we have*

$$\left|\sum_{n \leq x} e(\alpha S(n^2))\right| \leq 4q^{7/2} (\log q)^{5/2} \tau(q)^{1/2} \left(1 + \frac{\log x}{\log q}\right)^{(1/2)\omega(q)+4} x^{1-\sigma_q(\alpha)}.$$

In particular, this means that for x sufficiently large

$$\sum_{n \leq x} e(\alpha S(n^2)) \ll (\log x)^{(1/2)\omega(q)+4} x^{1-\sigma_q(\alpha)}.$$

However, some of Aloui's theorems do not follow from Theorem 1. For example, Aloui showed that, for a given $r \in \mathbb{N}, q$ and $m \geq 2$ such that $(m, q-1) = 1$ and $s > 0$ be a real number, we have

$$\sum_{n \leq x} \sum_{\substack{d \mid n \\ S(d^2) \equiv r \bmod m}} \left(\frac{n}{d}\right)^s = \frac{\frac{\zeta(s+1)}{s+1} + \beta}{m} x^{s+1} + \begin{cases} O(x \log x), & \text{if } s = 1, \\ O(x^t), & \text{if } s \neq 1, \end{cases}$$

as $x \to \infty$, where $t = \max(1, s)$ and

$$\beta = \sum_{j=1}^{m-1} e\left(\frac{-jr}{m}\right) \int_1^{\infty} \left(\sum_{k \leq t} e\left(\frac{jS(k^2)}{m}\right)\right) \frac{dt}{t^{s+2}}.$$

To this, he only used Abel's summation formula to prove the above result. On the same way, we shall extend the paper of Aloui to the summatory functions of the following arithmetical functions by independent Theorem 1. The number of nonisomorphic abelian groups with n elements under constraints on the sum of the digits of squares and the k-free numbers are considered in this paper. Let $t(n)$ be the number of nonisomorphic abelian groups with n elements.

The Dirichlet series for $t(n)$ is

$$\sum_{n=1}^{\infty} t(n)n^{-s} = \zeta(s)\zeta(2s)\zeta(3s)\zeta(4s)\cdots, \qquad \mathrm{Re}\,(s) > 1.$$

We know that

$$t(n) = \sum_{\delta|n} d(1,2,3;\delta)v(\frac{n}{\delta}),$$

where for $\mathrm{Re}\,(s) > \frac{1}{4}$,

$$V(s) = \sum_{n=1}^{\infty} v(n)n^{-s} = \zeta(4s)\zeta(5s)\zeta(6s)\cdots$$

so that $V(s)$ converges absolutely for $\mathrm{Re}\,(s) > \frac{1}{4}$ and thus

$$\sum_{n\leq x} |\ v(n)\ | \ll x^{\frac{1}{4}+\epsilon}. \tag{1}$$

We refer the reader to monograph [4] (see Chapter XIV, pp 413-415) for more details. For a given $r \in \mathbb{N}, q$ and $m \geq 2$ such that $(m, q-1) = 1$, we define a function that depend on n, q, r and m by

$$\bar{t}(n) = \sum_{\substack{\delta|n \\ S(\delta^2)\equiv r \bmod m}} d(1,2,3;\delta)v(\frac{n}{\delta}).$$

The first aim of this paper is to estimate

$$\bar{T}(x) = \sum_{n\leq x} \bar{t}(n), \qquad x > 1$$

and we obtain the following results

Theorem 2. *For a given $r \in \mathbb{N}, q$ and $m \geq 2$ such that $(m, q-1) = 1$. We have*

$$\bar{T}(x) = \frac{V(1)+y(2)}{m}\zeta(2)\zeta(3)x + \frac{2V(\frac{1}{2})+y(\frac{3}{2})}{2m}\zeta(\frac{1}{2})\zeta(\frac{3}{2})x^{\frac{1}{2}} + \frac{3V(\frac{1}{3})+y(\frac{4}{3})}{3m}\zeta(\frac{1}{3})\zeta(\frac{2}{3})x^{\frac{1}{3}} + O(x^{\frac{1}{4}+\epsilon}),$$

as $x \to \infty$, *where*

$$y(s) = \sum_{j=1}^{m-1} e\left(\frac{-jr}{m}\right) \int_1^{\infty} \left(\sum_{k \le t} v(k) e\left(\frac{jS(k^2)}{m}\right)\right) \frac{dt}{t^s}.$$

The proof of Theorem 1.2 relies on the three-dimensional divisor problem, which stated in the asymptotic formula (see in [4], pp 397-407)

$$\sum_{n \le x} d(a,b,c;n) = \zeta(b/a)\zeta(c/a)x^{1/a} + \zeta(a/b)\zeta(c/b)x^{1/b} + \zeta(a/c)\zeta(b/c)x^{1/c} + \Delta(a,b,c;x),$$

where $\Delta(a,b,c;x) \ll x^{\theta}$, for $\frac{1}{4} < \theta < \frac{1}{3}$.

Moreover, we consider the number of k-free integers ($k \ge 2$ fixed) not exceeding x, under a constraint of the sum of digit of squares. A natural number n is square-free if it is a product of different primes, and more generally n is called *k-free* ($k \ge 2$ fixed) if in the canonical decomposition $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ we have $\alpha_1 \le k-1, \ldots, \alpha_r \le k-1$. Therefore if $f_k(n) = 1$ when n is k-free and zero otherwise, then $f_k(n)$ is the characteristic function of the set of k-free numbers, and for $\mathrm{Re}\,(s) > 1$ we define

$$\sum_{n=1}^{\infty} f_k(n) n^{-s} = \prod_p \left(1 + p^{-s} + \cdots + p^{-(k-1)s}\right) = \prod_p \frac{1 - p^{-ks}}{1 - p^{-s}} = \frac{\zeta(s)}{\zeta(ks)}. \tag{2}$$

In particular,

$$f_2(n) = \mu^2(n) \qquad \text{and} \quad \mu^2(n) = \sum_{d^2 \mid n} \mu(d).$$

From the product representation (2) we see that $f_k(n)$, the characteristic function of the set of k-free numbers, may be expressed as

$$f_k(n) = \sum_{d^k \mid n} \mu(d).$$

Here, we set

$$\bar{f}_k(n) = \sum_{\substack{d^k \mid n \\ S(d^2) \equiv r \bmod m}} \mu(d)$$

and

$$G(n) = \begin{cases} \mu(l) & \text{if } n = l^k \text{ and } S(l^2) \equiv r \bmod m, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\bar{f}_k(n) = \sum_{d|n} G(d).$$

We wish to estimate the sum

$$\bar{F}(x) = \sum_{n \le x} \bar{f}_k(n), \qquad x > 1.$$

Theorem 3. *For a given $r \in \mathbb{N}, q$ and $m \ge 2$ such that $(m, q-1) = 1$, we have*

$$\bar{F}(x) = \frac{1}{m}\left(\frac{1}{\zeta(k)} + k\sum_{j=1}^{m-1} e\left(\frac{-rj}{m}\right)\rho_j\right)x + O(x^{1/k}),$$

for some absolute constant $C > 0$ and

$$\rho_j = \int_1^\infty \left(\sum_{d \le u} \mu(d) e\left(\frac{jS(d^2)}{m}\right)\right)\frac{du}{u^{k+1}}.$$

Our method of proof is elementary. We use the following lemma and Abel's summation formula.

Lemma 1. *For any positive integers m and n we have*

$$\frac{1}{m}\sum_{j=0}^{m-1} e\left(\frac{jn}{m}\right) = \begin{cases} 1 & \text{if } m|n \\ 0 & \text{otherwise .} \end{cases} \tag{3}$$

2. Proof of Theorem 2

Given x large enough, we have

$$\begin{aligned} \bar{T}(x) &= \sum_{n \le x} \sum_{\substack{\delta | n \\ S(\delta^2) \equiv r \bmod m}} d(1,2,3;\delta) v(\frac{n}{\delta}) \\ &= \sum_{\substack{\delta, l \\ \delta l \le x \\ S(\delta^2) \equiv r \bmod m}} d(1,2,3;\delta) v(\frac{n}{\delta}) \\ &= \sum_{\substack{k \le x \\ S(k^2) \equiv r \bmod m}} v(k) \sum_{n \le \frac{x}{k}} d(1,2,3;n) \end{aligned}$$

$$= \sum_{\substack{k\le x\\ S(k^2)\equiv r \bmod m}} v(k)\Bigg(\zeta(2)\zeta(3)\frac{x}{k}+\zeta(\frac{1}{2})\zeta(\frac{3}{2})\left(\frac{x}{k}\right)^{\frac{1}{2}} + \zeta(\frac{1}{3})\zeta(\frac{2}{3})\left(\frac{x}{k}\right)^{\frac{1}{3}}+\Delta(1,2,3;\frac{x}{k})\Bigg)$$

$$= \zeta(2)\zeta(3)x \sum_{\substack{k\le x\\ S(k^2)\equiv r \bmod m}} \frac{v(k)}{k} + \zeta(\frac{1}{2})\zeta(\frac{3}{2})x^{\frac{1}{2}} \sum_{\substack{k\le x\\ S(k^2)\equiv r \bmod m}} \frac{v(k)}{k^{\frac{1}{2}}}$$

$$+ \zeta(\frac{1}{3})\zeta(\frac{2}{3})x^{\frac{1}{3}} \sum_{\substack{k\le x\\ S(k^2)\equiv r \bmod m}} \frac{v(k)}{k^{\frac{1}{3}}} + \sum_{\substack{k\le x\\ S(k^2)\equiv r \bmod m}} v(k)\Delta(1,2,3;\frac{x}{k}).$$

In view of the bound of Δ, we obtain, for $\frac{1}{4}<\theta<\frac{1}{3}$, the last sum as the error term

$$\sum_{\substack{k\le x\\ S(k^2)\equiv r \bmod m}} v(k)\Delta(1,2,3;\frac{x}{k}) = O\left(\sum_{k\le x} \mid v(k) \mid x^{\theta}k^{-\theta}\right).$$

It remains only detecting the congruences. To this, we use the orthogonality relation (3). Then, we have

$$\sum_{\substack{k\le x\\ S(k^2)\equiv r \bmod m}} \frac{v(k)}{k} = \frac{1}{m}\sum_{k\le x}\frac{v(k)}{k}\sum_{j=0}^{m-1} e\left(\frac{j(S(k^2)-r)}{m}\right)$$

$$= \frac{1}{m}\sum_{j=0}^{m-1} e\left(\frac{-jr}{m}\right)\sum_{k\le x}\frac{v(k)}{k}e\left(\frac{jS(k^2)}{m}\right)$$

$$= \frac{1}{m}\sum_{k\le x}\frac{v(k)}{k} + \frac{1}{m}\sum_{j=1}^{m-1} e\left(\frac{-jr}{m}\right)\sum_{k\le x}\frac{v(k)}{k}e\left(\frac{jS(k^2)}{m}\right).$$

Using partial summation and (1) we find that

$$\sum_{k\le x}\frac{v(k)}{k} = V(1)+O(x^{\frac{-3}{4}+\epsilon}).$$

Now we let

$$U_j(x) = \sum_{k\le x}\frac{v(k)}{k}e\left(\frac{jS(k^2)}{m}\right).$$

Using Abel's summation formula, then, for $j = 1, 2, ..., m-1$

$$U_j(x) = \frac{1}{x}\sum_{k\leq x} v(k)e\left(\frac{jS(k^2)}{m}\right) + \int_1^x \left(\sum_{k\leq t} v(k)e\left(\frac{jS(k^2)}{m}\right)\right)\frac{dt}{t^2}.$$

Since the latter integral is convergent, we have

$$\int_1^x \left(\sum_{k\leq t} v(k)e\left(\frac{jS(k^2)}{m}\right)\right)\frac{dt}{t^2}$$
$$= \int_1^\infty \left(\sum_{k\leq t} v(k)e\left(\frac{jS(k^2)}{m}\right)\right)\frac{dt}{t^2} + O(x^{\frac{-3}{4}+\epsilon}),$$

since

$$\int_x^\infty \left(\sum_{k\leq t} v(k)e\left(\frac{jS(k^2)}{m}\right)\right)\frac{dt}{t^2} = O(|\int_x^\infty t^{\frac{-7}{4}+\epsilon}dt\ |) = O(x^{\frac{-3}{4}+\epsilon}).$$

Moreover, it is clear that

$$\frac{1}{x}\sum_{k\leq x} v(k)e\left(\frac{jS(k^2)}{m}\right) = O(x^{\frac{-3}{4}+\epsilon}).$$

Then, for $j = 1, 2, ..., m-1$,

$$U_j(x) = \int_1^\infty \left(\sum_{k\leq t} v(k)e\left(\frac{jS(k^2)}{m}\right)\right)\frac{dt}{t^2} + O(x^{\frac{-3}{4}+\epsilon}).$$

Then,

$$\sum_{\substack{k\leq x \\ S(k^2)\equiv r \bmod m}} \frac{v(k)}{k} = \frac{1}{m}V(1)$$
$$+ \frac{1}{m}\sum_{j=1}^{m-1} e\left(\frac{-jr}{m}\right)\int_1^\infty \left(\sum_{k\leq t} v(k)e\left(\frac{jS(k^2)}{m}\right)\right)\frac{dt}{t^2} + O(x^{\frac{-3}{4}+\epsilon}).$$

Similarly,

$$\sum_{\substack{k\leq x \\ S(k^2)\equiv r \bmod m}} \frac{v(k)}{k^{\frac{1}{2}}} = \frac{1}{m}V(\frac{1}{2})$$
$$+ \frac{1}{2m}\sum_{j=1}^{m-1} e\left(\frac{-jr}{m}\right)\int_1^\infty \left(\sum_{k\leq t} v(k)e\left(\frac{jS(k^2)}{m}\right)\right)\frac{dt}{t^{\frac{3}{2}}} + O(x^{\frac{-1}{4}+\epsilon})$$

and

$$\sum_{\substack{k\le x\\ S(k^2)\equiv r \bmod m}} \frac{v(k)}{k^{\frac{1}{3}}} = \frac{1}{m}V(\frac{1}{3}) + \frac{1}{3m}\sum_{j=1}^{m-1} e\left(\frac{-jr}{m}\right)\int_1^\infty \left(\sum_{k\le t} v(k)e\left(\frac{jS(k^2)}{m}\right)\right)\frac{dt}{t^{\frac{4}{3}}} + O(x^{\frac{-1}{12}+\epsilon}).$$

Now we let, for a real number s,

$$y(s) = \sum_{j=1}^{m-1} e\left(\frac{-jr}{m}\right)\int_1^\infty \left(\sum_{k\le t} v(k)e\left(\frac{jS(k^2)}{m}\right)\right)\frac{dt}{t^s}.$$

Then,

$$\begin{aligned}
\bar{T}(x) &= \zeta(2)\zeta(3)x\left(\frac{1}{m}(V(1)+y(2)) + O(x^{\frac{-3}{4}+\epsilon})\right)\\
&\quad + \zeta(\frac{1}{2})\zeta(\frac{3}{2})x^{\frac{1}{2}}\left(\frac{1}{m}(V(\frac{1}{2})+\frac{1}{2}y(\frac{3}{2})) + O(x^{\frac{-1}{4}+\epsilon})\right)\\
&\quad + \zeta(\frac{1}{3})\zeta(\frac{2}{3})x^{\frac{1}{3}}\left(\frac{1}{m}(V(\frac{1}{3})+\frac{1}{3}y(\frac{4}{3})) + O(x^{\frac{-1}{12}+\epsilon})\right)\\
&\quad + O\left(\sum_{k\le x} \mid v(k) \mid x^\theta k^{-\theta}\right), \text{ for } \quad \frac{1}{4} < \theta < \frac{1}{3}\\
&= \frac{V(1)+y_j(2)}{m}\zeta(2)\zeta(3)x\\
&\quad + \frac{2V(\frac{1}{2})+y(\frac{3}{2})}{m}\zeta(\frac{1}{2})\zeta(\frac{3}{2})x^{\frac{1}{2}}\\
&\quad + \frac{3V(\frac{1}{3})+y(\frac{4}{3})}{m}\zeta(\frac{1}{3})\zeta(\frac{2}{3})x^{\frac{1}{3}}\\
&\quad + O(x^{\frac{1}{4}+\epsilon}).
\end{aligned}$$

3. Proof of Theorem 3

Let $yz = x$.

$$\begin{aligned}
\bar{F}(x) &= \sum_{n\le x}\bar{f}_k(n) = \sum_{n\le x}\sum_{d\mid n} G(d) = \sum_{\substack{d,d_1\\ dd_1\le x}} G(d)\\
&= \sum_{d\le y} G(d)\sum_{d_1\le \frac{x}{d}} 1 + \sum_{d_1\le z}\sum_{d\le \frac{x}{d_1}} G(d) - \sum_{d\le y} G(d)\sum_{d_1\le z} 1
\end{aligned}$$

$$= S_1 + S_2 - S_3,$$

say. In view of (3), we have

$$S_1 = \sum_{d\le y} G(d) \sum_{d_1 \le \frac{x}{d}} 1 = \sum_{d\le y} G(d) \left\lfloor \frac{x}{d} \right\rfloor = \sum_{\substack{d\le y^{1/k} \\ S(d^2)\equiv r \bmod m}} \mu(d) \left\lfloor \frac{x}{d^k} \right\rfloor$$

$$= \frac{1}{m} \sum_{d\le y^{1/k}} \mu(d) \left\lfloor \frac{x}{d^k} \right\rfloor \sum_{j=0}^{m-1} e\left(\frac{j}{m}(S(d^2)-r)\right)$$

$$= \frac{1}{m} \sum_{d\le y^{1/k}} \mu(d) \left\lfloor \frac{x}{d^k} \right\rfloor + \frac{1}{m} \sum_{j=1}^{m-1} e\left(\frac{-rj}{m}\right) \sum_{d\le y^{1/k}} \mu(d) \left\lfloor \frac{x}{d^k} \right\rfloor e\left(\frac{jS(d^2)}{m}\right).$$

Since, for a real x, $\lfloor x \rfloor = x + O(1)$,

$$\sum_{d\le y^{1/k}} \mu(d) \left\lfloor \frac{x}{d^k} \right\rfloor = \sum_{d\le y^{1/k}} \mu(d) \left(\frac{x}{d^k} + O(1)\right)$$

$$= x \sum_{d\le y^{1/k}} \frac{\mu(d)}{d^k} + O(y^{1/k}).$$

We have

$$\sum_{d\le y^{1/k}} \frac{\mu(d)}{d^k} = \sum_{d=1}^{\infty} \frac{\mu(d)}{d^k} + O\left(\left|\sum_{d>y^{1/k}} \frac{\mu(d)}{d^k}\right|\right)$$

$$= \frac{1}{\zeta(k)} + O\left(\left|\int_{y^{1/k}}^{\infty} \frac{1}{t^k} dt\right|\right)$$

$$= \frac{1}{\zeta(k)} + O\left(y^{1/k-1}\right).$$

It remains to consider the second term; we set, for $j = 1, 2, ..., m-1$,

$$S_{1j} = \sum_{d\le y^{1/k}} \mu(d) \left\lfloor \frac{x}{d^k} \right\rfloor e\left(\frac{jS(d^2)}{m}\right).$$

Here, we apply Abel's summation formula to S_{1j} in order to obtain

$$S_{1j} = \sum_{d\le y^{1/k}} \mu(d) \left(\frac{x}{d^k} + O(1)\right) e\left(\frac{jS(d^2)}{m}\right)$$

$$= x \sum_{d\le y^{1/k}} \frac{\mu(d)}{d^k} e\left(\frac{jS(d^2)}{m}\right) + O(y^{1/k})$$

$$= x\left[\frac{1}{y}\sum_{d\le y^{1/k}}\mu(d)e\left(\frac{jS(d^2)}{m}\right)\right.$$
$$\left. + k\int_1^{y^{1/k}}\left(\sum_{d\le u}\mu(d)e\left(\frac{jS(d^2)}{m}\right)\right)\frac{du}{u^{k+1}}\right] + O(y^{1/k}).$$

The integral is convergent. Then, we obtain

$$\int_1^{y^{1/k}}\left(\sum_{d\le u}\mu(d)e\left(\frac{jS(d^2)}{m}\right)\right)\frac{du}{u^{k+1}} = \rho_j + O(y^{1/k-1}),$$

where

$$\rho_j = \int_1^{\infty}\left(\sum_{d\le u}\mu(d)e\left(\frac{jS(d^2)}{m}\right)\right)\frac{du}{u^{k+1}}.$$

It is clear, that

$$\frac{1}{y}\sum_{d\le y^{1/k}}\mu(d)e\left(\frac{jS(d^2)}{m}\right) = O(y^{1/k-1}).$$

Collecting all, we have, for $j = 1, 2, ..., m-1$,

$$S_{1j} = kx\rho_j + O(xy^{1/k-1}) + O(y^{1/k}),$$

and

$$S_1 = \frac{1}{m}\left(\frac{x}{\zeta(k)} + O(xy^{1/k-1}) + O(y^{1/k})\right)$$
$$+ \frac{1}{m}\sum_{j=1}^{m-1} e\left(\frac{-rj}{m}\right)\left(kx\rho_j + O(xy^{1/k-1}) + O(y^{1/k})\right)$$
$$= \frac{1}{m}\left(\frac{1}{\zeta(k)} + k\sum_{j=1}^{m-1} e\left(\frac{-rj}{m}\right)\rho_j\right)x + O(xy^{1/k-1}) + O(y^{1/k}).$$

Next we consider S_2

$$S_2 = \sum_{d_1\le z}\sum_{d\le\frac{x}{d_1}} G(d) = \sum_{d_1\le z}\sum_{\substack{d\le(\frac{x}{d_1})^{1/k}\\ S(d^2)\equiv r \bmod m}}\mu(d)$$
$$= \frac{1}{m}\sum_{d_1\le z}\sum_{d\le(\frac{x}{d_1})^{1/k}}\mu(d)\sum_{j=0}^{m-1} e\left(\frac{j}{m}(S(d^2)-r)\right)$$

$$= \frac{1}{m} \sum_{d_1 \leq z} \sum_{d \leq (\frac{x}{d_1})^{1/k}} \mu(d)$$
$$+ \frac{1}{m} \sum_{j=1}^{m-1} e\left(\frac{-jr}{m}\right) \sum_{d_1 \leq z} \sum_{d \leq (\frac{x}{d_1})^{1/k}} \mu(d) e\left(\frac{j}{m} S(d^2)\right).$$

To estimate the first sum, we need the fact (see [4]) that

$$\sum_{n \leq x} \mu(n) = O(xN(x)), \qquad \sum_{n \leq x} |\mu(n)| = O(x),$$

where

$$N(x) = \exp(-C \log^{3/5} x (\log\log)^{-1/5})),$$

for some absolute constant $C > 0$. It is clear that,

$$\sum_{d_1 \leq z} \sum_{d \leq (\frac{x}{d_1})^{1/k}} \mu(d) \ll x^{1/k} \sum_{d_1 \leq z} d_1^{1/k}$$
$$\ll x^{1/k} z^{1-1/k}.$$

To estimate the second sum, we write

$$\sum_{d_1 \leq z} \sum_{d \leq (\frac{x}{d_1})^{1/k}} \mu(d) e\left(\frac{j}{m} S(d^2)\right)$$
$$\ll \sum_{d_1 \leq z} \left(\left(\sum_{d \leq (\frac{x}{d_1})^{1/k}} |\mu(d)|^2 \right)^{1/2} \left(\sum_{d \leq (\frac{x}{d_1})^{1/k}} \left| e\left(\frac{j}{m} S(d^2)\right) \right|^2 \right)^{1/2} \right)$$
$$\ll \sum_{d_1 \leq z} \left(\frac{x}{d_1}\right)^{1/k}$$
$$\ll x^{1/k} z^{1-1/k}.$$

Thus,

$$S_2 \ll x^{1/k} z^{1-1/k}.$$

For S_3, we write

$$S_3 = \sum_{d \leq y} G(d) \sum_{d_1 \leq z} 1$$
$$= (\frac{x}{y} + O(1)) \sum_{\substack{d \leq y^{1/k} \\ S(d^2) \equiv r \bmod m}} \mu(d)$$

$$= O(xy^{1/k-1}).$$

Collecting all, we obtain

$$\bar{F}(x) = \frac{1}{m}\left(\frac{1}{\zeta(k)} + k\sum_{j=1}^{m-1} e\left(\frac{-rj}{m}\right)\rho_j\right)x + O(x^{1/k}z^{1-1/k} + xy^{1/k-1} + y^{1/k}).$$

We choose $x = y$ and $z = 1$. Now $0 < u < \frac{1}{k}$. Then the error term in later sum are $x^{1/k}$.

References

1. K. Aloui, *On the average of some arithmetical functions under a constraint on the sum of digits of squares*, Hiroshima Math. J. **46** (2016), no. 1, 37–54.
2. M. Drmota, C. Mauduit, and J. Rivat, *Primes with an average sum of digits*, Compos. Math. **145** (2009), 271–292.
3. A. O. Gelfond, *Sur les nombres qui ont des propriétés additives et multiplicatives données*, Acta Arith. **13** (1968), 259–265.
4. A. Ivić, *The theory of the Riemann zeta-function with applications*, John Wiley & Sons, New York, 1985.
5. C. Mauduit and J. Rivat, *La somme des chiffres des carrés*, Acta Math. **203** (2009), 107–148.
6. C. Mauduit and J. Rivat, *Sur un problème de Gelfond: la somme des chiffres des nombres premiers*, Ann. of Math **171** (2010), 1591–1646.
7. M. Mkaouar and W. Wannes, *On the number of restricted prime factors of an integer*, Acta Math. Hungar. **143** (2014), no. 1, 88–95.
8. J. Sándor and B. Crstici, *Handbook of number theory.* II, Kluwer Acad. Publ., Dordrecht, 2004.

Department of Mathematics, Faculty of Science, Kasetsart University, Bangkok, 10900, Thailand
Email: fscitrp@ku.ac.th

Voronoï's Impact on Modern Science, Book 6, Volume 1
Proceedings of the Sixth International Conference on Analytic Number Theory and Spatial Tessellations

ON THE UNIVERSALITY FOR ZETA-FUNCTION OF CERTAIN CUSP FORMS

Adelė Vaiginytė (*Lithuania*)

Abstract. In the paper, a variety of results on the universality theorems on the approximation of analytic functions by shifts of zeta-functions of Hecke-eigen cusp forms $\zeta(s, F)$ is briefly overviewed.

Introduction

The remarkable property of universality was discovered by Voronin [10] by analyzing the Riemann zeta-function. Potential of approximation of analytic functions by certain shifts of one and the same function gained attention of specialists. For instance, Reich [9], Gonek [2], Good [3], Bagchi [1], Laurinčikas [4, 5] and others improved and generalized the Voronin theorem for other functions given by Dirichlet series, including Dirichlet L function, and Dedekind, Hurwitz and Lerch zeta-functions. In this paper, some recent results on the universality for zeta-functions attached to certain cusp forms are reviewed.

1. Basic notions

Denote by $s = \sigma + it$ a complex variable. Let

$$SL(2, \mathbb{Z}) := \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

be the full modular group. It is well known that the transformation

$$\gamma(z) = \frac{az + b}{cz + d},$$

$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$, $\mathrm{Im}(z) > 0$, keeps the variable in the upper half-plane of the complex plane. Further, a holomorphic cusp form

Received August 15, 2018; revised November 20, 2018.
2010 *Mathematics Subject Classification.* Primary 11M41; Secondary 30K10.
Key words and phrases. Hecke-eigen cusp form, uniform distribution modulo 1, universality, zeta-function of cusp form.

is defined. The function $F(z)$, $z \in \mathbb{C}$, is called a holomorphic cusp form of weight κ for $SL(2,\mathbb{Z})$ if it is holomorphic for $\mathrm{Im}(z) > 0$, for all $\gamma \in SL(2,\mathbb{Z})$ satisfies the functional equation

$$F\left(\frac{az+b}{cz+d}\right) = (cz+d)^{\kappa}F(z)$$

and at infinity has the Fourier series expansion

$$F(z) = \sum_{m=1}^{\infty} c(m)e^{2\pi imz}.$$

Moreover, we assume that $F(z)$ is an eigen form of all Hecke operators

$$T_m F(z) = m^{\kappa-1} \sum_{\substack{a,d>0\\ad=m}} \frac{1}{d^{\kappa}} \sum_{b \pmod d} F\left(\frac{az+b}{d}\right), \quad m \in \mathbb{N}.$$

Then $c(m) \neq 0$, therefore, it can be normalized to have the Fourier coefficient $c(1) = 1$.

The zeta-function $\zeta(s,F)$ associated with the cusp form $F(z)$ of weight κ is defined, for $\sigma > \frac{\kappa+1}{2}$, by the Dirichlet series

$$\zeta(s,F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s},$$

and is analytically continued to an entire function. As some other zeta-functions, $\zeta(s,F)$ has the Euler product expansion over primes and the corresponding functional equation.

2. Universality

The first result on the universality of $\zeta(s,F)$ was obtained by Laurinčikas and Matsumoto in 2001 [6]. In this section, their result and several current implementations of the universality theorems for $\zeta(s,F)$ will be presented.

Let $D = D_F = \left\{s \in \mathbb{C} : \frac{\kappa}{2} < \sigma < \frac{\kappa+1}{2}\right\}$, $\mathcal{K} = \mathcal{K}_F$ be the class of compact subsets in the strip D with connected complements, and $H_0(K)$, $K \in \mathcal{K}$, stand for the class of continuous non-vanishing functions on K that are analytic in the interior of K. The Lebesgue measure of a measurable set $A \subset \mathbb{R}$ is denoted by $\mathrm{meas}\, A$. Then the Laurinčikas–Matsumoto universality theorem for $\zeta(s,F)$ can be formulated as follows.

Theorem 1. *Suppose that $K \in \mathcal{K}$, $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$, the following inequality*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau, F) - f(s)| < \varepsilon\right\} > 0$$

holds.

In Theorem 1, shifts τ in $\zeta(s+i\tau, F)$ are taken from a continuous set and could take any real values. Also, it was proved in [7] that with shifts from discrete sets analogous results can be attained.

Let $\#A$ denote the cardinality of the set A. Then the following is true.

Theorem 2. *Suppose that $K \in \mathcal{K}$, $f(s) \in H_0(K)$ and $h > 0$ is an arbitrary fixed number. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\{0 \le k \le N : \sup_{s\in K} |\zeta(s+ikh, F) - f(s)| < \varepsilon\} > 0.$$

Due to their forms, Theorem 1 is called a continuous universality theorem and Theorem 2 is called a discrete universality theorem.

Recently, some generalized results were obtained by Laurinčikas, Šiaučiūnas and the author [8]. The basic idea is that the shifts τ are taken from a more general set. In fact, a special class of functions is defined. We say that a function $\varphi(\tau) \in U(\tau_0)$, $\tau_0 > 0$, if the following conditions are satisfied:

(1) $\varphi(\tau)$ is a differentiable real-valued positive increasing function on $[\tau_0, \infty)$;
(2) $\varphi'(\tau)$ is monotonic and positive on $[\tau_0, \infty)$ satisfying $\frac{1}{\varphi'(\tau)} = o(\tau)$, $\tau \to \infty$;
(3) $\varphi(2\tau) \max_{\tau\le t\le 2\tau} \frac{1}{\varphi'(t)} \ll \tau$, $\tau \to \infty$.

Then, the following result is true.

Theorem 3. *Suppose that $\varphi(\tau) \in U(\tau_0)$, $K \in \mathcal{K}$, $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T-\tau_0} \operatorname{meas}\left\{\tau \in [\tau_0, T] : \sup_{s\in K} |\zeta(s+i\varphi(\tau), F) - f(s)| < \varepsilon\right\} > 0.$$

An analogous result for the discrete case requires slightly different conditions. We remind that a sequence $\{x_k : k \in \mathbb{N}\} \subset \mathbb{R}$ is called uniformly distributed modulo 1 if, for every $I = [a, b) \subset [0, 1)$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_I(\{x_k\}) = b - a,$$

where $\{x_k\}$ denotes the fractional part of x_k, and χ_I is the indicator function of set I.

We say that $\varphi(t) \in U(k_0)$, $k_0 \in \mathbb{N}$, if:

(1) $\varphi(t)$ is a differentiable real-valued positive increasing function on $t \in [k_0 - \frac{1}{2}, \infty)$;
(2) $\varphi'(t)$ is increasing and positive on $t \in [k_0 - \frac{1}{2}, \infty)$, satisfying $\varphi(2t) \max_{t \le u \le 2t} \frac{1}{\varphi'(u)} \ll t$, $t \to \infty$,
(3) $\{a\varphi(k) : k \ge k_0\}$ is uniformly distributed modulo 1 for every $a \in \mathbb{R} \setminus \{0\}$.

Theorem 4. *Suppose $\varphi(t) \in U(k_0)$, $K \in \mathcal{K}$, $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N \to \infty} \frac{1}{N - k_0 + 1} \#\{k_0 \le k \le N : \sup_{s \in K} |\zeta(s + i\varphi(k), F) - f(s)| < \varepsilon\} > 0.$$

References

1. B. Bagchi, *The statistical behaviour and universality properties of the Riemann zeta-function and other allied Dirichlet series*, PhD Thesis, Indian Statistical Institute, Calcutta, 1981.
2. S. M. Gonek, *Analytic properties of zeta and L-functions*, Math. Z. **181** (1982), 319–334.
3. A. Good, *On the distribution of values of Riemann's zeta-function*, Act. Arith. **38** (1981), 347–388.
4. A. Laurinčikas, *The universality theorem*, Lith. Math. J. **23** (1983), 283–289. (in Russian)
5. A. Laurinčikas, *The universality theorem.* II, Lith. Math. J. **24** (1984), 143–149. (in Russian)
6. A. Laurinčikas and K. Matsumoto, *The universality of zeta-functions attached to certain cusp forms*, Acta Arith. **98** (2001), 345–359.
7. A. Laurinčikas, K. Matsumoto, and J. Steuding, *Discrete universality of L-functions of new forms.* II, Lith. Math. J. **56** (2016), no. 2, 207–218.
8. A. Laurinčikas, D. Šiaučiūnas, and A. Vaiginytė, *Extension of the discrete universality theorem for zeta-functions of certain cusp forms*, Nonlinear Anal. Model. Control **23** (2018), no. 6, 961–973.
9. A. Reich, *Zur Universalität und Hypertranszendenz der Dedekindschen Zetafunktion*, Abh. Braunschweig. Wiss. Ges. **33** (1982), 197–203.
10. S. M. Voronin, *Theorem on the "universality" of the Riemann zeta-function*, Izv. Akad. Nauk SSSR Ser. Mat. **39** (1975), 475–486. (in Russian).

Faculty of Mathematics and Informatics, Vilnius University, 24 Naugarduko St., Vilnius, LT-03225, Lithuania
Email: adele.vaiginyte@mif.vu.lt

Voronoï's Impact on Modern Science, Book 6, Volume 1
Proceedings of the Sixth International Conference on Analytic Number Theory and Spatial Tessellations

PROBLEM OF ELLIPSE IN ARITHMETIC PROGRESSIONS

Pavel Varbanets and Sergey Varbanets (*Ukraine*)

Abstract. We study the distribution of solutions of the congruence relation $u^2 + dv^2 \equiv \ell \pmod D$ with growing module D in narrow sectors. This problem generalizes the circle problem for arithmetic progressions.

Introduction

In [5], the problem of the number $A(x, D)$ of solutions of the congruence relation $x^2+y^2 \equiv 1 \pmod D$ in the circle $u^2+v^2 \le x$ was considered, and the asymptotic formula for $A(x, D)$, being non-trivial for $D \le x^{\frac{2}{3}-\varepsilon}$, where $\varepsilon > 0$ is an arbitrary small number, was obtained. This problem is an analog of the divisor problem for arithmetic progressions studied by C. Hooley [3]:

$$uv \equiv 1 \pmod D,\ uv \le x, \tag{1}$$

for which the non-triviality region is analogous, i.e., $D \le x^{\frac{2}{3}-\varepsilon}$.

In the present paper, we study the distribution of solutions of the congruence relation

$$\begin{gathered} u^2 + dv^2 \equiv 1 \pmod D, \\ d > 0 \text{ is a square-free number},\ (d, D) = 1, \end{gathered} \tag{2}$$

under the condition for a point $(u, v\sqrt{d})$ to belong to a narrow sectorial elliptic region $u^2 + dv^2 \le x$.

We use the following notation: $\mathbb{Q}(\sqrt{-d})$ is the imaginary quadratic field; $K := \left\{a + b\sqrt{-d} \mid a, b \in \mathbb{Z}\right\}$; g is the number of elements e_0 in K with norm $N(e_0) = 1$; $s = \sigma + it$ is the complex variable, $\Re(s) = \sigma$, $\Im(s) = t$; for $\omega = a + b\sqrt{-d} \in K$, we write $N(\omega) = a^2 + db^2$; for $\gamma \in K$, K_γ denotes the complete residue system modulo γ of K.

Received August 15, 2018; revised November 20, 2018.
2010 *Mathematics Subject Classification.* 11L05, 11L07, 11N25.
Key words and phrases. Hecke Z-function, exponential sum, number of solutions of the congruence relation, narrow sector.

1. Auxiliary results

Denote, by $Z_m(s;\delta_1,\delta_2)$, the series

$$\sum_{\substack{\omega\in K\\ \omega\neq-\delta_1}}\frac{e^{gmi\arg\omega}}{N(\omega+\delta_1)^s}e^{2\pi i\Re(\delta_2\omega)}.$$

This series is absolutely convergent for $\sigma>1$ and is called the Epstein series of the quadratic form u^2+dv^2 with shift (δ_1,δ_2). Note that, for all $\omega\in K$ and every unit e_0 $(N(e_0)=1)$, we have

$$e^{gmi\arg e_0\omega}=e^{gmi\arg\omega}.$$

Lemma 1. *If $m\neq0$, then $Z_m(s;\delta_1,\delta_2)$ is an entire function. If $m=0$, then, for $\delta_2\notin K$, $Z_m(s;\delta_1,\delta_2)$ is still an entire function. For $m=0$ and $\delta_2\in K$, the function $Z_m(s;\delta_1,\delta_2)$ is holomorphic, except for a simple pole at the point $s=1$ with residue $\dfrac{\pi}{\sqrt{d}}$. Moreover, the functional equation*

$$(\pi d)^{-s}\Gamma\left(\frac{g|m|}{2}+s\right)Z_m(s;\delta_1,\delta_2)=$$
$$=(\pi d)^{-(1-s)}\Gamma\left(\frac{g|m|}{2}+1-s\right)Z_m(1-s;-\delta_2,\delta_1)e^{-2\pi i\Re(\delta_1\delta_2)}$$

holds.

The proof of this lemma runs in the same way as for $d=1$ (see, for details, [1]).

Now, let

$$\Psi_m(s)=\pi^{1-2s}\frac{\Gamma\left(\frac{g|m|}{2}+1-s\right)}{\Gamma\left(\frac{g|m|}{2}+s\right)}.$$

It is not difficult to see that, for $-1\le\sigma\le2$,

$$\Psi_m(s)\ll(MT)^{1-2\sigma}+(MT)^{-\sigma},$$

where $M=|m|+3$ and $T=|t|+3$.

Lemma 2 (see [4], Lemma 8)**.** *Let $y\ge1$, $k=0,1,2$, $a\in\mathbb{R}$, $-1<a\le\dfrac{5}{4}$, $\eta(a)=\min j=0,1,2|a-j|\neq0$. Then, for any real numbers u and v, the estimate*

$$\int_{a+iu}^{a+iv}\frac{y^s\Psi_m(s)}{s(s+1)\cdots(s+k)}ds\ll M^{1-2a}y^a\left(\eta^{-1}(a)+\log M\right)+y^{\frac12-\frac{2k+1}{4}}$$

holds with the absolute constant in the symbol "$\ll$".

Lemma 3. *Let p be a prime number, $(p,d)=1$, and let m be a positive integer. Then, for $\omega \in K$, we have*

$$E := \max_{\substack{n \\ (n,p)=1}} \left| \sum_{\substack{\tau \in K_{p^m} \\ N(\tau) \equiv n \pmod{p^m}}} e^{2\pi i \Re\left(\frac{\omega\tau}{p^m}\right)} \right| \le (\nu+1) p^{\frac{m+\nu}{2}},$$

where $\nu = \max\limits_{0 \le \ell \le m} \left(p^\ell \mid \omega\right)$.

Proof. Let $\omega = p^\ell(c_1 + c_2\sqrt{-d})$, $(c_1, c_2, p) = 1$. Since $(n,p)=1$, we have

$$\tau = u + v\sqrt{-d},\ (u,v,p) = 1$$

and

$$\Re(\omega\tau) = a_1 u + a_2 v,\ (a_1, a_2, p^m) = p^\nu.$$

Thus, with regard for the estimate of the Gaussian sum

$$\left| \sum_{u \in \mathbb{Z}_{p^m}} e^{2\pi i \frac{Au^2+Bv}{p^m}} \right| \ll \begin{cases} 0 & if \quad \nu_p(B) < \nu_p(A) \le m, \\ p^{\frac{m+m_0}{2}} & if \quad A = A + 0p^m,\ (A_0, p) = 1, \\ & \qquad \nu_p(B) \ge \nu_p(A) = m_0, \end{cases}$$

(here, for $a \in \mathbb{Z}$, $\nu_p(a)$ denotes a p-adic order, i.e., $\nu_p(a) = k$ if $p^k | a$ and $p^{k+1} \not| a$) and for the Kloosterman sum

$$\left| \sum_{\substack{h \in \mathbb{Z}^*_{p^m} \\ hh' \equiv 1 \pmod{p^m}}} e^{2\pi i \frac{ah+bh'}{p^m}} \right| \ll 2p^{\frac{m}{2}} \ \ if\ (a,b,p) = 1,$$

we obtain the assertion of the lemma after simple calculations. □

2. Main results

Let $r(n,d)$ denote the number of representations of n in the form $n = u^2 + dv^2$, $u, v \in \mathbb{Z}$. We put

$$A(x, D) = \sum_{\substack{n \equiv 1 \pmod{D} \\ n \le x}} r(n,d).$$

First, we prove the fundamental lemma. We use the notation

$$\Omega(n) = \sum_{\substack{u_1,u_2 \\ u_1^2+du_2^2=n}} \Phi(D;u_1,u_2),$$

$$a_n = \int_0^\infty e^{-u}u^{\frac{1}{2}s}I_1\left(\frac{2\pi}{D}\sqrt{nx}u^{\frac{1}{2}s}\right)du$$

and

$$\Phi(D;u_1,u_2) = \sum_{\substack{\ell_1,\ell_2 \pmod D \\ {\ell_1}^2+d\ell_2^2\equiv 1 \pmod D}} e^{2\pi i\frac{u_1\ell_1+u_2\ell_2}{D}},$$

where $I_1(z)$ is the Bessel function of the first kind.

Lemma 4. *For $x>0$, $s>0$, and every $\varepsilon>0$, we have*

$$R_d(x,D) := A(x,D) - \frac{\pi x}{\sqrt{d}D^2} \sum_{\substack{\ell_1,\ell_2 \pmod D \\ \ell_1^2+d\ell_2^2\equiv 1 \pmod D}} 1 =$$

$$= \frac{x^{\frac{1}{2}}}{D}\sum_{n=1}^\infty \frac{\Omega(n)}{\sqrt{n}}a_n + O\left(\frac{x^{1+\varepsilon}}{sD}\right) + O\left(x^\varepsilon\right),$$

where the constants with symbol "O" depend only on ε.

Proof. The application of the Poisson summation formula for the function

$$f(u,v) = \exp\left(-\left(\frac{(\ell_1+Du)^2+d(\ell_2+Dv)^2}{x}\right)^s\right)$$

gives

$$\sum_{p,q=-\infty}^\infty \exp\left\{-\left(\frac{(\ell_1+Dp)^2+d(\ell_2+Dq)^2}{x}\right)^s\right\} =$$

$$= \sum_{p,q=-\infty}^\infty \int_{-\infty}^\infty\int_{-\infty}^\infty \left(e^{2\pi i(pu+qv)}\times \right.$$

$$\left. \times \exp\left\{-\left(\frac{(\ell_1+Du)^2+d(\ell_2+Dv)^2}{x}\right)^s\right\}\right)dudv =$$

$$= \frac{1}{2} \cdot \frac{x\sqrt{d}}{D^2} \sum_{p,q=-\infty}^{\infty} \left(\int_0^{\infty} e^{-R^s} dR \times \right.$$

$$\times \int_0^{2\pi} \left(e^{-2\pi i \frac{p\ell_1 + q\ell_2}{D}} \cos\left(2\pi p \frac{\sqrt{Rxd}}{D} \cos\varphi \right) \cdot \right.$$

$$\left. \left. \cdot \cos\left(2\pi q \frac{\sqrt{Rx}}{D} \sin\varphi \right) \right) d\varphi \right) =$$

$$= \frac{\pi x\sqrt{d}}{D^2} \Gamma\left(1 + \frac{1}{s}\right) +$$

$$+ \frac{\pi x\sqrt{d}}{D^2} \sum_{p,q=-\infty}^{\infty} e^{2\pi i \frac{p\ell_1 + q\ell_2}{D}} \int_0^{\infty} e^{-R^s} I_0\left(\frac{2\pi}{D} \sqrt{(p^2 + dq^2)Rx} \right) dR = \tag{3}$$

$$= \frac{\pi x\sqrt{d}}{D^2} \Gamma\left(1 + \frac{1}{s}\right) +$$

$$+ \frac{\pi x\sqrt{d}}{D^2} \sum_{n=1}^{\infty} \sum_{\substack{u,v \\ u^2+dv^2=n}} e^{2\pi i \frac{u\ell_1 + v\ell_2}{D}} \int_0^{\infty} e^{-R^s} I_0\left(\frac{2\pi}{D} \sqrt{nxR} \right) dR =$$

$$= \frac{\pi x\sqrt{d}}{D^2} \left(1 + O\left(\frac{1}{s}\right)\right) +$$

$$+ \frac{s\sqrt{xd}}{D} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sum_{\substack{u,v \\ u^2+dv^2=n}} e^{2\pi i \frac{u\ell_1 + v\ell_2}{D}} \int_0^{\infty} e^{-R^s} R^{s-\frac{1}{2}} I_1\left(\frac{2\pi}{D} \sqrt{nxR} \right) dR =$$

$$= \frac{\pi x\sqrt{d}}{D^2} \left(1 + O\left(\frac{1}{s}\right)\right) +$$

$$+ \frac{\sqrt{xd}}{D} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sum_{\substack{u,v \\ u^2+dv^2=n}} e^{2\pi i \frac{u\ell_1 + v\ell_2}{D}} \int_0^{\infty} e^{-u} u^{\frac{1}{2s}} I_1\left(\frac{2\pi}{D} \sqrt{nx} u^{\frac{1}{2s}} \right) du.$$

Now, we make use an idea by V. Brun of the approximation of the function

$$F(x) = \begin{cases} 1 & if \quad 0 \le x < 1, \\ 0 & if \quad x > 1. \end{cases}$$

We take $\exp\{-x^s\}$ as an approximating function. Then we obtain

$$A(x, D) =$$

$$= \sum_{\substack{\ell_1,\ell_2 \pmod D \\ {\ell_1}^2+\ell_2^2\equiv 1 \pmod D}} \sum_{p,q=-\infty}^{\infty} \exp\left\{-\left(\frac{(\ell_1+Dp)^2+(\ell_2+Dq)^2}{x}\right)^s\right\} =$$

$$= \sum_{\substack{\ell_1,\ell_2 \pmod D \\ {\ell_1}^2+d\ell_2^2\equiv 1 \pmod D}} \sum_{\substack{u_1\equiv\ell_1 \pmod D \\ u_2\equiv\ell_2 \pmod D \\ u_1^2+du_2^2\le x}} \left(1-\exp\left\{-\left(\frac{u_1^2+du_2^2}{x}\right)^s\right\}\right) -$$

$$- \sum_{\substack{\ell_1,\ell_2 \pmod D \\ {\ell_1}^2+d\ell_2^2\equiv 1 \pmod D}} \sum_{\substack{u_1\equiv\ell_1 \pmod D \\ u_2\equiv\ell_2 \pmod D \\ u_1^2+du_2^2> x}} \exp\left\{-\left(\frac{u_1^2+du_2^2}{x}\right)^s\right\} =$$

$$= \sum_{\substack{n\equiv 1 \pmod D \\ n\le x}} r(n,d)\left(1-\exp\left(-\left(\frac{n}{x}\right)^s\right)\right) - \tag{4}$$

$$- \sum_{\substack{n\equiv 1 \pmod D \\ n> x}} r(n,d)\exp\left(-\left(\frac{n}{x}\right)^s\right) =$$

$$= O\left(\int_0^{\frac{x}{D}} (uD)^\varepsilon \left(\frac{uD}{x}\right)^s du\right) +$$

$$+ O\left(\int_{\frac{x}{D}}^{\infty} (uD)^\varepsilon \exp\left(-\left(\frac{uD}{x}\right)^s\right) du\right) + O\left(x^\varepsilon\right) =$$

$$= O\left(\frac{x^{1+\varepsilon}}{sD}\right) + O\left(x^\varepsilon\right).$$

Estimates (3) and (4) prove the lemma. □

Theorem 1. *Let $Y > 1$. Then*

$$R_d(x,D) = \frac{\sqrt{xd}}{D}\int_{x^{-1}}^{x} e^{-u}u^{\frac{1}{2s}} \sum_{1\le n\le Y} \frac{\Omega(n)}{\sqrt{n}} I_1\left(\frac{2\pi}{D}\sqrt{nx}u^{\frac{1}{2s}}\right) du +$$

$$+ O\left(\frac{x^{1+\varepsilon}\sqrt{d}}{sD}\right) + O\left(x^\varepsilon\sqrt{d}\right) + O\left(\frac{sD^{1+\varepsilon}}{x^{\frac{1}{4}}}Y^{-\frac{1}{4}+\varepsilon\sqrt{d}}\right) +$$

$$+ O\left(\frac{sD^{2+\varepsilon}}{x^{\frac{3}{4}}}Y^{-\frac{3}{4}+\varepsilon\sqrt{d}}\right) + O\left(\frac{sD^{\frac{1}{2}+\varepsilon}}{x}\sqrt{d}\log Y\right) +$$

$$+ O\left(\frac{D^\varepsilon}{x}Y^{\frac{1}{4}}\sqrt{d}\right).$$

Proof. In the expression for a_n, we substitute u^{2s} for u and integrate by parts. We find

$$\left|\frac{a_n}{\sqrt{d}}\right| = \frac{sD}{\pi\sqrt{nx}}\left|\int\limits_0^\infty I_0\left(\frac{2\pi}{D}\sqrt{nx}u\right)\frac{d}{du}\left(e^{-u^{2s}}u^{2s}\right)du\right| \ll$$
$$\ll \frac{sD^{\frac{3}{2}}}{(nx)^{\frac{3}{4}}}\int\limits_0^\infty\left(u^{-\frac{1}{2}}+\frac{D}{\sqrt{nx}}u^{-\frac{3}{2}}\right)\frac{d}{du}\left(e^{-u^{2s}}u^{2s}\right)du \ll \tag{5}$$
$$\ll \frac{sD^{\frac{3}{2}}}{(nx)^{\frac{3}{4}}}+\frac{sD^{\frac{5}{2}}}{(nx)^{\frac{5}{4}}}.$$

Let q^2 be the greatest quadratic divisor of the greatest common divisor of the numbers n and D. Then, for any representation of $n = u_1^2 + u_2^2$, we have $(u_1, u_2, D)|q$. Therefore, Lemma 4 yields

$$\Omega(n) = \sum_{u_1^2+u_2^2=n}\prod_{p|D}\Phi\left(p^m; u_1, u_2\right) \ll r(n,d)\prod_{p|D}c^{\cdot\frac{m}{2}}\left(u_1, u_2, p^m\right)^{\frac{1}{2}} \ll$$
$$\ll r(n,d)\prod_{p|D}c^{\cdot\frac{m}{2}}\left(q, p^m\right)^{\frac{1}{2}} \ll r(n,d)(Dq)^{\frac{1}{2}}c^{\nu(D)} \ll \tag{6}$$
$$\ll r(n,d)q^{\frac{1}{2}}D^{\frac{1}{2}+\varepsilon},$$

where $\nu(D)$ is the number of prime divisors of the number D. Since

$$\nu(D) \ll \frac{\log D}{\log\log D},$$

we have $c^{\nu(D)} \ll D^{\varepsilon}$. Therefore, in view of (5), we have

$$\sum_{n>Y}\frac{\Omega(n)}{\sqrt{n}}a_n \ll \frac{sD^{\frac{3}{2}}}{x^{\frac{3}{4}}}\sum_{q^2|D}\sum_{nq^2>Y}\frac{q^{\frac{1}{2}}D^{\frac{1}{2}+\varepsilon}r(nq^2,d)}{q^{\frac{5}{2}}n^{\frac{5}{4}}}\left(1+\frac{D}{\sqrt{nq^2}x}\right) \ll$$
$$\ll \frac{sD^{2+\varepsilon}}{x^{\frac{3}{4}}}\left(Y^{-\frac{1}{4}+\varepsilon}+\frac{D}{x^{\frac{1}{2}}}Y^{-\frac{3}{4}+\varepsilon}\right)\sum_{q^2|D}\frac{1}{q^{\frac{3}{2}-\varepsilon}} \ll \tag{7}$$
$$\ll \frac{sD^{2+\varepsilon}}{x^{\frac{3}{4}}}\left(Y^{-\frac{1}{4}+\varepsilon}+\frac{D}{x^{\frac{1}{2}}}Y^{-\frac{3}{4}+\varepsilon}\right).$$

Further,

$$\frac{x^{\frac{1}{2}}}{D}\sum_{1\le n\le Y}\frac{|\Omega(n)|}{\sqrt{n}}\left|\int\limits_0^{x^{-1}} e^{-u}u^{\frac{1}{2s}}I_1\left(\frac{2\pi}{D}\sqrt{nx}u^{\frac{1}{2s}}\right)du\right| =$$

$$= \frac{x^{\frac{1}{2}}}{D}\sum_{1\le n\le Y}\frac{|\Omega(n)|}{\sqrt{n}}2se^{-x^{-1}}x^{-1}\left|\int\limits_0^{x^{-\frac{1}{2s}}} I_1\left(\frac{2\pi}{D}\sqrt{nx}z\right)dz\right| \ll \tag{8}$$

$$\ll \frac{sD^{\frac{1}{2}+\varepsilon}}{x}\sum_{1\le n\le Y}\frac{r(n,d)}{n} \ll \frac{sD^{\frac{1}{2}+\varepsilon}}{x}\log Y.$$

Finally,

$$\frac{x^{\frac{1}{2}}}{D}\sum_{1\le n\le Y}\frac{|\Omega(n)|}{\sqrt{n}}\left|\int\limits_x^{\infty} e^{-u}u^{\frac{1}{2s}}I_1\left(\frac{2\pi}{D}\sqrt{nx}u^{\frac{1}{2s}}\right)du\right| \ll$$

$$\ll \sum_{1\le n\le Y}\frac{|\Omega(n)|}{n^{\frac{3}{4}}}\left(\frac{x^{\frac{1}{4}}}{D^{\frac{1}{2}}}+\frac{D^{\frac{1}{2}}}{n^{\frac{1}{2}}x^{\frac{1}{4}}}\right)\int\limits_x^{\infty} e^{-u}udu \ll \tag{9}$$

$$\ll \frac{D^{\varepsilon}}{x}Y^{\frac{1}{4}}+\frac{D^{1+\varepsilon}}{x^{\frac{3}{2}}}Y^{-\frac{1}{4}}.$$

The assertion of the lemma follows from (7)–(9). □

If $s = \dfrac{x^{\frac{1}{2}}}{D^{\frac{3}{4}}}$ and $Y = \dfrac{D^2}{x}$, then Theorem 1 immediately yields the following proposition.

Theorem 2. *For $D^{\frac{3}{2}} \le x < D^2$, the asymptotic formula*

$$A(x,D) = \frac{\pi x}{\sqrt{d}D}\gamma_0\prod_{p|D}\left(1-\left(\frac{d}{p}\right)\cdot\frac{1}{p}\right)+$$

$$+O\left(D^{\frac{1}{2}}\tau(D)\exp\left(c\frac{(\ln D)^{\frac{1}{2}}}{\ln\ln D}\right)\right)+O\left(\frac{x^{\frac{1}{2}}}{D^{\frac{1}{4}}}\tau(D)\right)$$

holds, where

$$\gamma_0 = \begin{cases} 1 & if \quad D \not\equiv 0 \pmod 4, \\ 2 & if \quad D \equiv 0 \pmod 4, \end{cases}$$

and $\tau(D)$ is the number of divisors of $D \in \mathbb{N}$.

3. Integral points of an ellipse in narrow sectors

In this section, we suppose that $\mathbb{Q}(\sqrt{-d})$ is a one-class field. Hecke [2] studied the zeta-function $Z(s,\Xi)$, which, for $\Re(s)>1$, is defined by the absolutely convergent series

$$Z(s;\Xi)=\sum_{\omega\in K}\frac{\Xi(\omega)}{N(\omega)^s},$$

where $\Xi(\omega)=e^{gmi\arg\omega}\cdot\chi(\omega)$, and χ is the group character modulo γ of the multiplicative group K_γ^*.

The Hecke Z-function $Z(s,\Xi)$ is an entire function, except for the case $m=0$ and the principal character χ_0 modulo γ. For the primitive character Ξ, Hecke [2] obtained the functional equation

$$Z(s,\Xi)=\kappa(\Xi)\Psi_m(s)Z(1-s,\overline{\Xi}), \tag{10}$$

where

$$\kappa(\Xi)=i^{-mg_0}(N(\gamma))^{-\frac{1}{2}}\chi(-1)e^{img_0\arg\gamma}\sum_{\alpha\in K_\gamma}\chi(\alpha)e^{2\pi i\Re\left(\frac{\alpha}{\gamma}\right)}$$

and

$$\Psi_m(s)=\left(\frac{1}{\pi}(N(\gamma))^{\frac{1}{2}}\right)^{1-2s}\frac{\Gamma\left(\frac{g|m|}{2}+1-s\right)}{\Gamma\left(\frac{g|m|}{2}+s\right)}. \tag{11}$$

We will define a group character χ by

$$\chi(\omega)=\chi_D(N(\omega)),$$

where, for $\gamma=D\in\mathbb{N}$, χ_D is the Dirichlet character modulo D. So,

$$\Xi(\omega)=e^{gmi\arg\omega}\chi_D(N(\omega)).$$

Then, denoting

$$r_m(n)=\sum_{u^2+dv^2=n}e^{gmi\arg(u+iv)},$$

we find

$$\begin{aligned}&\sum_{n\le x}r_m(n)\chi_D(n)\sum_{n\le x}\sum_{u^2+dv^2=n}e^{gmi\arg(u+iv)}\chi_D(u+iv)=\\&=\sum_{N(\omega)}\Xi(\omega).\end{aligned} \tag{12}$$

Lemma 5. *Let $m\neq 0$, $-\frac{1}{2}\le\sigma_1\le 0$ and $1\le\sigma_2\le 2$. Then, for*

$$\sigma_1-\log^{-1}MT\le\sigma\le\sigma_2+\log^{-1}MT,\ \ (M=|m|+3,\ T=|t|+3),$$

we have

$$\sum_{\chi_D} Z(s,\Xi) \ll D^{1-2\sigma_1\frac{\sigma_2-\sigma}{\sigma_2-\sigma_1}+\log^{-1}MT} E^{\frac{\sigma_2-\sigma}{\sigma_2-\sigma_1}} (MT)^{\frac{1-2\sigma_1}{\sigma_2-\sigma_1}(\sigma_2-\sigma)} \log^2(DMT),$$

where

$$E = \max_{\substack{n\\(n,D)=1}} \left| \sum_{\substack{\alpha\in K_D\\ N(\alpha)\equiv n \pmod D}} e^{2\pi i\Re\left(\frac{\alpha}{D}\right)} \right|.$$

With the application of the Phragmén–Lindelöf principle, the lemma is an easy corollary of the estimates for $Z(s,\Xi)$ on the lines

$$\Re(s) = \sigma_1 - \frac{1}{\log MT} \quad \text{and} \quad \Re(s) = \sigma_2 + \frac{1}{\log MT}.$$

Corollary. *For $m \neq 0$, the estimate*

$$\sum_{\chi_D} Z(0,\Xi) \ll EM \log M \cdot \tau(D)$$

holds.

Theorem 3. *For $m \neq 0$ and $D \ll x \ll D^2$, the bound*

$$\sum_{\substack{n\equiv\ell \pmod D\\ n\leq x}} r_m(n) \ll D^{\frac12}\log x + \frac{x^{\frac12}}{D^{\frac12}}\log M + D^{\frac12} M \log M \cdot \tau(D)$$

holds.

Proof. The function $Z(s,\Xi)$ is entire for $m \neq 0$. Thus, we have

$$\begin{aligned} \frac{1}{2\pi i}\int\limits_{(2)} \frac{x^{s+1}Z(s,\Xi)}{s(s+1)(s+2)}ds = \\ = \frac{x^2}{2}Z(0,\Xi) + \frac{1}{2\pi i}\int\limits_{(-\frac14)} \frac{x^{s+2}Z(s,\Xi)}{s(s+1)(s+2)}ds. \end{aligned} \tag{13}$$

Denote

$$S(x,\Xi) = \frac12 \sum_{0<N(\omega)\leq x} \Xi(\omega)(x-N(\omega))^2.$$

Then we have

$$\frac{1}{2\pi i}\int\limits_{(2)}\frac{x^{s+1}Z(s,\Xi)}{s(s+1)(s+2)}ds=$$

$$=\sum_{\omega}\Xi(\omega)N(\omega)^2\cdot\frac{1}{2\pi i}\int(2)\frac{\left(\frac{x}{N(\omega)}\right)^{s+2}}{s(s+1)(s+2)}ds= \tag{14}$$

$$=\frac{1}{2}\sum_{0<N(\omega)\le x}\Xi(\omega)(x-N(\omega))^2=S(x,\Xi).$$

For the primitive character Ξ in view of the functional equation (10), relations (13) and (14) yield

$$S(x,\Xi)=\frac{x^2}{2}Z(o,\Xi)+\kappa(\Xi)\sum_{\omega}\overline{\Xi}(\omega)N(\omega)^{-3}W(xN(\omega)), \tag{15}$$

where

$$W(y)=\frac{1}{2\pi i}\int\limits_{(-\frac{1}{4})}\frac{y^{s+2}\Psi_m(s)}{s(s+1)(s+2)}ds.$$

Consider the operation on the set of twice continuously differentiable functions on $[0,\infty)$:

$$\Delta_z F(x)=\sum_{j=0}^{2}(-1)^j\begin{pmatrix}2\\ j\end{pmatrix}F(x+jz)=\int\limits_{x}^{x+z}dy_1\int\limits_{y_1}^{y_1+z}F''(y_2)dy_2.$$

Then we have

$$\Delta_z\left(\frac{x^2}{2}Z(0,\Xi)\right)=z^2Z(0,\Xi). \tag{16}$$

It is clear that

$$W(y)=\frac{1}{2\pi i}\int\limits_{(b)}\frac{y^{s+2}\Psi_m(s)}{s(s+1)(s+2)}ds$$

for any $b\in(-1,0)$. Thus, by Lemma 2 for $b=-1+\dfrac{1}{\log y}$, we get

$$W(y)\ll M^3y(\log y+\log M)\cdot D^3:=K(y,M). \tag{17}$$

Hence,

$$\Delta_z W(xN(\omega))\ll K(xN(\alpha),M)$$

if and only if the inequality $z \ll N(\omega)$ holds.

The value $\Delta_z W(xN(\omega))$ can be estimated in another way. For this purpose, consider $W''(y)$. Let

$$\Phi(y) = \frac{1}{2\pi i} \int\limits_{\left(\frac{5}{4}\right)} \frac{y^{s+2}\Psi_m(s)}{s(s+1)(s+2)} ds.$$

Then

$$\Phi(y) = \frac{y^2}{2}\Psi_m(0) + \frac{1}{2\pi i} \int\limits_{\left(\frac{5}{4}\right)} \frac{y^{s+2}\Psi_m(s)}{s(s+1)(s+2)} ds = \frac{y^2}{2}\Psi_m(0) + W(y).$$

Hence, it follows that

$$W''(y) = -\Psi_m(0) + \frac{1}{2\pi i} \int\limits_{\left(\frac{5}{4}\right)} \frac{y^s}{s}\Psi_m(s) ds.$$

The shift of the contour in (17) to the line $\Re(s) = (\log y)^{-1}$ and the application of Lemma 2 give

$$W''(y) = DM \log M + y^{\frac{1}{4}} := L(y, M).$$

Therefore, we have

$$\Delta_z W(xN(\omega)) \ll z^2 N^2(\omega) L(xN(\omega)M). \tag{18}$$

Hence, for $D = p^k$, p is prime (thus, it is indecomposable in K), we deduce

$$\begin{aligned} S(x,m) := \sum_{\chi_{p^k}} S(x,\Xi) = \frac{x^2}{2} \sum_{\chi_{p^k}} Z(0,\Xi) + \\ + \sum_{chi_{p^k}} \kappa(\Xi) \sum_{\omega} \overline{\Xi}(\omega) N(\omega)^{-s} W(x, N(\omega)) = \frac{x^2}{2} \sum_{\chi_{p^k}} Z(0,\Xi) + \\ + \sum_{\omega} \frac{e^{-gmi \arg \omega}}{N(\omega)} \cdot \frac{W(xN(\omega))}{\pi p^k} \sum_{\delta \in K_{p^k}} e^{2\pi i \Re\left(\frac{\delta}{p^k}\right)} \sum_{\chi_{p^\ell}} \chi(\overline{\ell} N(\delta\overline{\omega})). \end{aligned} \tag{19}$$

Now, we apply the operator Δ_z to equality (19). By the definition of $\kappa(\Xi)$, we obtain

$$\begin{aligned} \Delta_z S(x,m) = z^2 \sum_{\chi_{p^k}} Z(0,\Xi) + \sum_{\omega} e^{-gmi \arg \omega} \frac{\Delta_z W(xN(\omega))}{N(\omega)^3} \cdot \frac{1}{\pi p^k} \times \\ \times \sum_{\delta \in K_{p^k}} e^{2\pi i \Re\left(\frac{\delta}{p^k}\right)} \sum_{\chi_{p^k}} \chi_{p^k}(N(\delta)) \overline{\chi}(\ell N(\omega)). \end{aligned} \tag{20}$$

In the above sum, we split the summation interval into two parts:

$$N(\omega) \le x \quad \text{and} \quad N(\omega) > x.$$

Note that, for $N(\omega) > x$, we may take $z \ll x^2$. Let $z = p^k M \ll x^2$. Then, in view of inequalities (17) and (18), we find

$$\Delta_z S(x,m) \ll z^2 E k p^k M \log M,$$

where

$$E = \max_{n \in \mathbb{Z}_{p^k}^*} \left| \sum_{\substack{\delta \\ N(\delta) \equiv \ell n \pmod{p^k}}} e^{2\pi i \frac{\Re(\delta)}{p^k}} \right|.$$

Due to the fact that

$$S''(x,m) = \sum_{\chi_{p^k}} \sum_{\substack{\omega \\ N(\omega) \le x}} \Xi(\omega) := H(x,m), \tag{21}$$

the equality

$$\Delta_z S(x,m) = \int_x^{x+z} dy_1 \int_{y_1}^{y_1+z} H(y_2,m)\,dy_2, \tag{22}$$

and the inequality

$$|H(y_2,m) - H(x,m)| \le \varphi\left(p^k\right) \left| \sum_{\substack{x < N(\omega) \le y_2 \\ N(\omega) \equiv \ell \pmod{p^k}}} e^{gmi \arg \omega} \right|,$$

we easily derive (by Lemma 3)

$$\begin{gathered} H(y_2,m) = H(x,m) + \\ + O(z) + O\left(x^{\frac{1}{2}} p^{\frac{k}{2}} \log x \cdot \log p^k\right) + O\left(p^{\frac{3}{2}k} \log x\right). \end{gathered} \tag{23}$$

Hence, from (21)–(22), we get

$$\begin{aligned} &z^2 \left(H(x,m) + O(z) + O\left(x^{\frac{1}{2}} p^{\frac{k}{2}} \log x \log p^k\right) + O\left(p^{\frac{3}{2}k} \log x\right)\right) = \\ &= O\left(z^2 k p^k E M \log M\right). \end{aligned}$$

This means that the theorem is proved for $D = p^k$.

The general case can be proved analogously with the relation connecting $Z(s,\Xi)$ and $Z(s,\Xi^*)$, where Ξ is induced by the primitive character Ξ^*. □

Let $d > 1$ be a positive square-free integer. Then, for any $D > 1$, $(d, D) \neq 1$, $A(x; \varphi_1; \varphi_2; D)$ denotes the number of solutions of the congruence relation

$$u^2 + dv^2 \equiv \ell \pmod{D},\ (\ell, D) = 1$$

under the conditions

$$u^2 + dv^2 \leq x,\ \ \varphi_1 \leq \arg(u + v\sqrt{-d}) \leq \varphi_2.$$

Theorems 2 and 3 lead to the following assertion.

Theorem 4. *For $0 < \varphi_2 - \varphi_1 \leq \dfrac{\pi}{2}$, the asymptotic formula*

$$A(x; \varphi_1; \varphi_2; D) = \frac{(\varphi_2 - \varphi_1)x}{2\sqrt{d}D}\gamma_0 \prod_{p|D}\left(1 - \left(\frac{d}{p}\right)\cdot\frac{1}{p}\right) + \\ + O\left(D^{\frac{1}{2}}\tau(D)\exp\left(c\frac{(\ln D)^{\frac{1}{2}}}{\ln\ln D}\right)\right) + O\left(\frac{x^{\frac{1}{2}}}{D^{\frac{1}{4}}}\tau(D)\right),$$

where the constants with symbols "O" depend on d and $\varepsilon > 0$, holds.

References

1. L. Balyas and P. Varbanets, *Quadratic residues of the norm group in sectorial domains*, Algebra Discrete Math. **22** (2016), no. 2, 153–170.
2. E. Hecke, *Eine neue Art von Zetafunktionen und ihre Beziehungen zur Verteilung der Primzahlen.* I–II, Math. Z. **1** (1918), 357–376; **6** (1920), 11–51.
3. C. Hooley, *An asymptotic formula in the theory of numbers*, Proc. London Math. Soc. **7** (1957), 396–413.
4. J. Kubilius, *On one problem of multidimensional analytic number theory*, Proc. Vilnius Univ. **4** (1955), 5–41 (in Lithuanian).
5. P. Varbanets, *Problem of circle in arithmetic progression*, Mat. Zametki **8** (1970), no. 6, 787–798 (in Russian).

Department of Computer Algebra and Discrete Mathematics, Odessa I. I. Mechnikov National University, 2 Dvorianska St., Odessa, 65026, Ukraine
Email: varb@sana.od.ua

Department of Computer Algebra and Discrete Mathematics, Odessa I. I. Mechnikov National University, 2 Dvorianska St., Odessa, 65026, Ukraine
Email: varb@sana.od.ua

Voronoï's Impact on Modern Science, Book 6, Volume 1
Proceedings of the Sixth International Conference on Analytic Number Theory and Spatial Tessellations

SOLVING A FUNCTIONAL EQUATION INVOLVING ONE MULTIPLICATIVE AND TWO q-ADDITIVE FUNCTIONS

Boonrod Yuttanan*, Vichian Laohakosol,
and Sukrawan Mavecha (*Thailand*)

Abstract. Let $\mathbb{N}_0$ be the set of non-negative integers. For an integer $q \geq 2$, a function $f : \mathbb{N}_0 \to \mathbb{C}$ is said to be q-additive if

$$f(0) = 0 \qquad \text{and} \qquad f(n) = \sum_{r=0}^{\infty} f(a_r(n)q^r),$$

where $n = \sum_{r=0}^{\infty} a_r(n)q^r$ with $a_r(n) \in \{0, 1, \ldots, q-1\}$. In this work, we solve functional equations of the form

$$f(n) = g(n) + h(n) \qquad (n \in \mathbb{N}),$$

where f is a completely multiplicative function, g is a p-additive function and h is a q-additive function for some positive integers p and q. The solutions are of three types: the first type has f as the identity function, the second type has f as a Dirichlet character mod p, and third type has f as a Dirichlet character mod q.

Introduction

Let q be a fixed integer ≥ 2. Every $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ can be uniquely represented through the base q-representation of the form

$$n = \sum_{r=0}^{\infty} a_r(n)q^r$$

with $a_r(n) \in \{0, 1, \ldots, q-1\}$ and $a_r(n) = 0$ if $q^r > n$. An arithmetic function $f : \mathbb{N}_0 \longrightarrow \mathbb{C}$ is said to be q-additive if $f(0) = 0$ and

$$f(n) = \sum_{r=0}^{\infty} f(a_r(n)q^r)$$

holds for all $n = \sum_{r=0}^{\infty} a_r(n)q^r$. Let A_q be the set of all q-additive functions.

Received July 24, 2018; revised November 16, 2018.
2010 *Mathematics Subject Classification.* Primary 11K65; Secondary 11N37.
Key words and phrases. Completely multiplicative function, q-additive function.
*Corresponding author.

The notion of q-additive functions was first introduced by Gelfond in 1968 [1] to establish asymptotic formulas of certain quantities. In 1999, Uchida [6] showed that, for integers $p, q \geq 2$ such that $(\log p)/\log q$ is irrational, if $f \in A_p \cap A_q$, then there exist ℓ, m with $d = \gcd(p^\ell, q^m)$ such that $f(nd) = nf(d)$ for each $n \in \mathbb{N}$. In particular, if $\gcd(p, q) = 1$ and $f \in A_p \cap A_q$, then $f(n) = f(1)n$ for all $n \in \mathbb{N}$. In 2003, Puchta and Spilker [4] characterized those functions $f \in A_p \cap A_q$ subject to the conditions $p \nmid q$ and $q \nmid p$, for fixed integers $p, q \geq 2$. They proved that f can be uniquely written as a linear combination of step-functions and certain periodic functions. Their work was made slightly more explicit later by Seekam, Mavecha and Laohakosol in 2017 [5], i.e., they derived explicit shapes of functions $f \in A_p \cap A_q$ and a basis of the vector space of all functions $f \in A_p \cap A_q$ based on prime factorizations of p and q.

Let M^* be the set of all completely multiplicative functions and χ_p be a Dirichlet character $\pmod p$. In 2015, Kátai and Phong [2] proved that if $f \in M^* \cap A_p$ where p is an integer ≥ 2, then $f(n) = n$ or $f(n) = \chi_q(n)$ for $q \mid p$. Later, in 2017 [3], they determined all solutions of the functional equation $f(n) = g(n) + h(n)$, where $f \in M^*, g \in A_2$ and $h \in A_3$. They also found all $f \in M^*$ and $g \in A_q$ for which $f(n) = g^2(n)$.

In this paper motivated by [3], we solve functional equations of the form $f(n) = g(n) + h(n)$ where $f \in M^*$, $g \in A_p$ and $h \in A_q$ for $(p, q) \in \{(2, 5), (3, 4), (3, 5), (4, 5)\}$. Our main result reads:

Theorem 1. *If $f \in M^*$, $g \in A_p$ and $h \in A_q$, where*

$$(p, q) \in \{(2, 5), (3, 4), (3, 5), (4, 5)\},$$

satisfy

$$f(n) = g(n) + h(n) \qquad (n \in \mathbb{N}), \tag{1}$$

then

$$f(1) = g(1) + h(1) = 1,$$

and for every $n \in \mathbb{N}$, one of the following statements hold:

(1)	$f(n) = n$,	$g(n) = g(1)n$,	$h(n) = h(1)n$;
(2)	$f(n) = \chi_p(n)$,	$g(n) = \chi_p(n) - h(1)n$,	$h(n) = h(1)n$;
(3)	$f(n) = \chi_q(n)$,	$g(n) = g(1)n$,	$h(n) = \chi_q(n) - g(1)n$.

Proofs

Adopting the notation in the introduction, we start with two lemmas needed in the proof of the main theorem. The first lemma, whose proof is omitted, is a direct consequence of the result in [4].

Lemma 1 ([6])**.** *Let p and q be relatively prime integers ≥ 2. If $f \in A_p \cap A_q$, then $f(n) = f(1)n$ for all $n \in \mathbb{N}$.*

The next lemma is a generalization of [3, Lemma 3].

Lemma 2. *Let p and q be distinct integers greater than 1, let $N \in \mathbb{N}$, and let $f \in M^*, g \in A_p$ and $h \in A_q$. If $N \equiv k \pmod{p}$ and $N \equiv l \pmod{q}$, where $k \in \{1, 2, \ldots, p-1\}$ and $l \in \{1, 2, \ldots, q-1\}$, then*

$$f(N) = f(N-1) + g(k) - g(k-1) + h(l) - h(l-1).$$

Proof. It follows from (1) that

$$\begin{aligned} f(N) &= g(N) + h(N) = g(k) + g(N-k) + h(l) + h(N-l) \\ &= g(k) + g(N-1) - g(k-1) + h(l) + h(N-1) - h(l-1) \\ &= f(N-1) + g(k) - g(k-1) + h(l) - h(l-1). \end{aligned}$$

□

Proof of Theorem 1 We proceed by induction. First, we will show that the equation (1) has solutions of the above form (1)–(3) for positive integers $n \leq 11$.

Case $(p, q) = (2, 5)$. Let

$$L_{2,5} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 18, 20, 21, 25, 28\}.$$

Consider the following system of 20 equations

$$f(n) = g(n) + h(n), \qquad \text{for } n \in L_{2,5}. \tag{2}$$

Since $f \in M^*$, $g \in A_2$ and $h \in A_5$, the system (2) can be represented in terms of the unknown values

$$\begin{aligned} &f(2), f(3), f(5), f(7), f(11), \\ &g(1), g(2), g(4), g(8), g(16), \\ &h(1), h(2), h(3), h(4), h(5), h(10), h(15), h(20), h(25). \end{aligned}$$

Solving this system using the computer software "Mathematica", there are six solutions as shown in Table 1. Note that in the table, $i = \sqrt{-1}$.

In case (a), (1) holds for every $n \leq 11$.

In case (b), we see that $f(n) = \chi_2(n)$ and $h(n) = h(1)n$; hence, (2) holds for every $n \leq 11$.

For cases (c)–(f), we have $f(n) = \chi_5(n)$ and $g(n) = g(1)n$, so (3) holds for every $n \leq 11$.

Case $(p, q) = (3, 4)$. Let

$$L_{3,4} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 18, 21, 22, 24\}.$$

Solving the system of 19 equations for $f(n) = g(n)+h(n)$, where $f \in M^*$, $g \in A_3$, $h \in A_4$ and $n \in L_{3,4}$, we obtain seven solutions as shown in Table 2.

In case (a), (1) holds for every $n \leq 11$.

TABLE 1. Case $(p, q) = (2, 5)$

	(a)	(b)	(c)	(d)	(e)	(f)
$f(2)$	2	0	$-i$	i	1	-1
$f(3)$	3	1	i	$-i$	1	-1
$f(5)$	5	1	0	0	0	0
$f(7)$	7	1	$-i$	i	1	-1
$f(11)$	11	1	1	1	1	1
$g(2)$	$2g(1)$	$-2h(1)$	$2g(1)$	$2g(1)$	$2g(1)$	$2g(1)$
$g(4)$	$4g(1)$	$-4h(1)$	$4g(1)$	$4g(1)$	$4g(1)$	$4g(1)$
$g(8)$	$8g(1)$	$-8h(1)$	$8g(1)$	$8g(1)$	$8g(1)$	$8g(1)$
$g(16)$	$16g(1)$	$-16h(1)$	$16g(1)$	$16g(1)$	$16g(1)$	$16g(1)$
$h(2)$	$2h(1)$	$2h(1)$	$-i - 2g(1)$	$i - 2g(1)$	$1 - 2g(1)$	$-1 - 2g(1)$
$h(3)$	$3h(1)$	$3h(1)$	$i - 3g(1)$	$-i - 3g(1)$	$1 - 3g(1)$	$-1 - 3g(1)$
$h(4)$	$4h(1)$	$4h(1)$	$-1 - 4g(1)$	$-1 - 4g(1)$	$1 - 4g(1)$	$1 - 4g(1)$
$h(5)$	$5h(1)$	$5h(1)$	$-5g(1)$	$-5g(1)$	$-5g(1)$	$-5g(1)$
$h(10)$	$10h(1)$	$10h(1)$	$-10g(1)$	$-10g(1)$	$-10g(1)$	$-10g(1)$
$h(15)$	$15h(1)$	$15h(1)$	$-15g(1)$	$-15g(1)$	$-15g(1)$	$-15g(1)$
$h(20)$	$20h(1)$	$20h(1)$	$-20g(1)$	$-20g(1)$	$-20g(1)$	$-20g(1)$
$h(25)$	$25h(1)$	$25h(1)$	$-25g(1)$	$-25g(1)$	$-25g(1)$	$-25g(1)$

TABLE 2. Case $(p, q) = (3, 4)$

	(a)	(b)	(c)	(d)	(e)
$f(2)$	2	1	-1	0	0
$f(3)$	3	0	0	1	-1
$f(5)$	5	1	-1	1	1
$f(7)$	7	1	1	1	-1
$f(11)$	11	1	-1	1	-1
$g(2)$	$2g(1)$	$1 - 2h(1)$	$-1 - 2h(1)$	$2g(1)$	$2g(1)$
$g(3)$	$3g(1)$	$-3h(1)$	$-3h(1)$	$3g(1)$	$3g(1)$
$g(6)$	$6g(1)$	$-6h(1)$	$-6h(1)$	$6g(1)$	$6g(1)$
$g(9)$	$9g(1)$	$-9h(1)$	$-9h(1)$	$9g(1)$	$9g(1)$
$g(18)$	$18g(1)$	$-18h(1)$	$-18h(1)$	$18g(1)$	$18g(1)$
$h(2)$	$2h(1)$	$2h(1)$	$2h(1)$	$-2g(1)$	$-2g(1)$
$h(3)$	$3h(1)$	$3h(1)$	$3h(1)$	$1 - 3g(1)$	$-1 - 3g(1)$
$h(4)$	$4h(1)$	$4h(1)$	$4h(1)$	$-4g(1)$	$-4g(1)$
$h(8)$	$8h(1)$	$8h(1)$	$8h(1)$	$-8g(1)$	$-8g(1)$
$h(16)$	$16h(1)$	$16h(1)$	$16h(1)$	$-16g(1)$	$-16g(1)$

In case (b)–(c), we see that $f(n) = \chi_3(n)$ and $h(n) = h(1)n$; hence, (2) holds for every $n \leq 11$.

For cases (d)–(e), we have $f(n) = \chi_4(n)$ and $g(n) = g(1)n$, so (3) holds for every $n \leq 11$.

TABLE 3. Case $(p, q) = (3, 5)$

	(a)	(b)	(c)	(d)	(e)	(f)	(g)
$f(2)$	2	1	-1	$-i$	i	1	-1
$f(3)$	3	0	0	i	$-i$	1	-1
$f(5)$	5	1	-1	0	0	0	0
$f(7)$	7	1	1	$-i$	i	1	-1
$f(11)$	11	1	-1	1	1	1	1
$g(2)$	$2g(1)$	$1-2h(1)$	$-1-2h(1)$	$2g(1)$	$2g(1)$	$2g(1)$	$2g(1)$
$g(3)$	$3g(1)$	$-3h(1)$	$-3h(1)$	$3g(1)$	$3g(1)$	$3g(1)$	$3g(1)$
$g(6)$	$6g(1)$	$-6h(1)$	$-6h(1)$	$6g(1)$	$6g(1)$	$6g(1)$	$6g(1)$
$g(9)$	$9g(1)$	$-9h(1)$	$-9h(1)$	$9g(1)$	$9g(1)$	$9g(1)$	$9g(1)$
$g(18)$	$18g(1)$	$-18h(1)$	$-18h(1)$	$18g(1)$	$18g(1)$	$18g(1)$	$18g(1)$
$g(27)$	$27g(1)$	$-27h(1)$	$-27h(1)$	$27g(1)$	$27g(1)$	$27g(1)$	$27g(1)$
$h(2)$	$2h(1)$	$2h(1)$	$2h(1)$	$-i-2g(1)$	$i-2g(1)$	$1-2g(1)$	$-1-2g(1)$
$h(3)$	$3h(1)$	$3h(1)$	$3h(1)$	$i-3g(1)$	$-i-3g(1)$	$1-3g(1)$	$-1-3g(1)$
$h(4)$	$4h(1)$	$4h(1)$	$4h(1)$	$-1-4g(1)$	$-1-4g(1)$	$1-4g(1)$	$1-4g(1)$
$h(5)$	$5h(1)$	$5h(1)$	$5h(1)$	$-5g(1)$	$-5g(1)$	$-5g(1)$	$-5g(1)$
$h(10)$	$10h(1)$	$10h(1)$	$10h(1)$	$-10g(1)$	$-10g(1)$	$-10g(1)$	$-10g(1)$
$h(15)$	$15h(1)$	$15h(1)$	$15h(1)$	$-15g(1)$	$-15g(1)$	$-15g(1)$	$-15g(1)$
$h(20)$	$20h(1)$	$20h(1)$	$20h(1)$	$-20g(1)$	$-20g(1)$	$-20g(1)$	$-20g(1)$
$h(25)$	$25h(1)$	$25h(1)$	$25h(1)$	$-25g(1)$	$-25g(1)$	$-25g(1)$	$-25g(1)$

TABLE 4. Case $(p, q) = (4, 5)$

	(a)	(b)	(c)	(d)	(e)	(f)	(g)
$f(2)$	2	0	0	$-i$	i	1	-1
$f(3)$	3	1	-1	i	$-i$	1	-1
$f(5)$	5	1	1	0	0	0	0
$f(7)$	7	1	-1	$-i$	i	1	-1
$f(11)$	11	1	-1	1	1	1	1
$g(2)$	$2g(1)$	$-2h(1)$	$-2h(1)$	$2g(1)$	$2g(1)$	$2g(1)$	$2g(1)$
$g(3)$	$3g(1)$	$1-3h(1)$	$-1-3h(1)$	$3g(1)$	$3g(1)$	$3g(1)$	$3g(1)$
$g(4)$	$4g(1)$	$-4h(1)$	$-4h(1)$	$4g(1)$	$4g(1)$	$4g(1)$	$4g(1)$
$g(8)$	$8g(1)$	$-8h(1)$	$-8h(1)$	$8g(1)$	$8g(1)$	$8g(1)$	$8g(1)$
$g(12)$	$12g(1)$	$-12g(1)$	$-12g(1)$	$12g(1)$	$12g(1)$	$12g(1)$	$12g(1)$
$g(16)$	$16g(1)$	$-16g(1)$	$-16g(1)$	$16g(1)$	$16g(1)$	$16g(1)$	$16g(1)$
$h(2)$	$2h(1)$	$2h(1)$	$2h(1)$	$-i-2g(1)$	$i-2g(1)$	$1-2g(1)$	$-1-2g(1)$
$h(3)$	$3h(1)$	$3h(1)$	$3h(1)$	$i-3g(1)$	$-i-3g(1)$	$1-3g(1)$	$-1-3g(1)$
$h(4)$	$4h(1)$	$4h(1)$	$4h(1)$	$-1-4g(1)$	$-1-4g(1)$	$1-4g(1)$	$1-4g(1)$
$h(5)$	$5h(1)$	$5h(1)$	$5h(1)$	$-5g(1)$	$-5g(1)$	$-5g(1)$	$-5g(1)$
$h(10)$	$10h(1)$	$10h(1)$	$10h(1)$	$-10g(1)$	$-10g(1)$	$-10g(1)$	$-10g(1)$
$h(15)$	$15h(1)$	$15h(1)$	$15h(1)$	$-15g(1)$	$-15g(1)$	$-15g(1)$	$-15g(1)$
$h(20)$	$20h(1)$	$20h(1)$	$20h(1)$	$-20g(1)$	$-20g(1)$	$-20g(1)$	$-20g(1)$
$h(25)$	$25h(1)$	$25h(1)$	$25h(1)$	$-25g(1)$	$-25g(1)$	$-25g(1)$	$-25g(1)$

Case $(p,q)=(3,5)$. Let

$$L_{3,5}=\{1,2,3,4,5,6,7,8,9,10,11,12,\\ 14,15,16,18,20,21,22,24,25,27,28\}.$$

Solving the system of 23 equations for $f(n)=g(n)+h(n)$, where $f\in M^*$, $g\in A_3$, $h\in A_5$ and $n\in L_{3,5}$, there are seven solutions as shown in Table 3.

In case (a), (1) holds for every $n\le 11$.

In case (b)–(c), we see that $f(n)=\chi_3(n)$ and $h(n)=h(1)n$; hence, (2) holds for every $n\le 11$.

For cases (d)–(g), we have $f(n)=\chi_5(n)$ and $g(n)=g(1)n$, so (3) holds for every $n\le 11$.

Case $(p,q)=(4,5)$. Let

$$L_{4,5}=\{1,2,3,4,5,6,7,8,9,11,12,14,15,16,18,20,21,22,24,25,27,28\}.$$

Solving the system of 22 equations for $f(n)=g(n)+h(n)$, where $f\in M^*$, $g\in A_4$, $h\in A_5$ and $n\in L_{4,5}$, there are seven solutions as shown in Table 4.

In case (a), (1) holds for every $n\le 11$.

In case (b)–(c), we see that $f(n)=\chi_4(n)$ and $h(n)=h(1)n$; hence, (2) holds for every $n\le 11$.

For cases (d)–(g), we have $f(n)=\chi_5(n)$ and $g(n)=g(1)n$, so (3) holds for every $n\le 11$.

Finally, we verify the inductive step for all cases. Let N be a positive integer greater than 11.

• Suppose $f(n)=n$, $g(n)=g(1)n$ and $h(n)=h(1)n$ for all $n<N$. If N is not a prime number, then $f(N)=N$. If N is prime, then by Lemma 2, there exist integers k,l such that $N\equiv k \pmod p$ with $1\le k\le p-1$, $N\equiv l \pmod q$ with $1\le l\le q-1$ and

$$\begin{aligned} f(N) &= f(N-1)+g(k)-g(k-1)+h(l)-h(l-1)\\ &= (N-1)+g(1)k-g(1)(k-1)+h(1)l-h(1)(l-1)\\ &= (N-1)+g(1)+h(1)=(N-1)+f(1)=N. \end{aligned}$$

Therefore, $f(n)=\chi_p(n)$ for all positive integers n.

• Suppose $f(n)=\chi_p(n)$, $h(n)=h(1)n$ and $g(n)=\chi_p(n)-h(1)n$ for all $n<N$. Then it is obvious that $f(N)=\chi_p(N)$ if N is not prime. Assume N is prime. If $N\equiv k \pmod p$ and $N\equiv l \pmod q$, where $k\in\{1,2,\dots,p-1\}$ and $l\in\{1,2,\dots,q-1\}$, then by Lemma 2

$$f(N)=f(N-1)+g(k)-g(k-1)+h(l)-h(l-1)$$

$$\begin{aligned}&= \chi_p(N-1) + \chi_p(k) - h(1)k - \chi_p(k-1) + h(1)(k-1)\\ &\quad + h(1)l - h(1)(l-1)\\ &= \chi_p(k-1) + \chi_p(k) - \chi_p(k-1) = \chi_p(k) = \chi_p(N).\end{aligned}$$

Hence, $f(n) = \chi_p(n)$ for all positive integers n. Since $h \in A_q$ and

$$h = f - g = \chi_p - g \in A_p,$$

by Lemma 1, $h(n) = h(1)n$ for all n. Consequently, $g(n) = \chi_p(n) - h(1)n$ for all n.

• The remaining case $f(n) = \chi_q(n)$ can be obtained in a similar manner. □

It is conceivable, but we have not yet been able to show, that in general, the functional equation $f(n) = g(n) + h(n)$ where $f \in M^*, g \in A_p$ and $h \in A_q$ with $\gcd(p, q) = 1$ has solutions of the form (1)–(3). Clearly, if we can find all solutions for the basis step consisting of all small values of n, then our inductive step can be applied to obtain solutions for all $n \in \mathbb{N}$.

References

1. A. O. Gelfond, *Sur les nombres qui ont des propriétés additives et multiplicatives données* [*On numbers that have additive and multiplicative properties given*], Acta Arith. **13** (1968), 259–265.
2. I. Kátai and B. M. Phong, *Some relations among arithmetical functions*, Ann. Univ. Sci. Budap. Rolando Eötvös, Sect. Comput. **44** (2015), 49–57.
3. I. Kátai and B. M. Phong, *Some relations among multiplicative and q-additive functions*, Lith. Math. J. **57** (2017), no. 1, 30–37.
4. J.-C. Puchta and J. Spilker, *Functions that are both g- and h-additive*, Arch. Math. **89** (2003), 264–270.
5. N. Seekam, V. Laohakosol, and S. Mavecha, *Explicit determination of functions which are additive with respect to two parameters*, APAM Conference, Chulalongkorn University, Thailand, 2018.
6. Y. Uchida, *On p and q-additive functions*, Tokyo J. Math. **22** (1999), no. 1, 83–97.

Algebra and Applications Research Unit, Department of Mathematics and Statistics, Faculty of Science, Prince of Songkla University, Songkhla, 90110, Thailand
Email: boonrod.y@psu.ac.th

Department of Mathematics, Faculty of Science, Kasetsart University, Bangkok, 10900, Thailand
Email: fscivil@ku.ac.th

Department of Mathematics, Faculty of Science, King Mongkut's Institute of Technology Ladkrabang, Bangkok, 10520, Thailand
Email: sukrawan.ta@kmitl.ac.th

Voronoï's Impact on Modern Science, Book 6, Volume 1
Proceedings of the Sixth International Conference on Analytic Number Theory and Spatial Tessellations

ON THE VORONOI SUMMABILITY OF FOURIER INTEGRALS AND ITS CONJUGATE INTEGRALS

Liliya Boytsun and Tamara Rybnikova (*Ukraine*)

Abstract. We obtain sufficient conditions imposed on the function $p(t)$ and on the function $\lambda(t)$, under which the Fourier integral of the function $f(t) \in L(-\infty, \infty)$ is summable by the Voronoi method to $f(x)$, and its conjugate integral is summable by this method to $g(x) = \frac{1}{\pi} \int\limits_0^\infty \frac{f(x+t)-f(x-t)}{t}\, dt$.

1. The functional summability of integrals has been considered by G.F. Voronoi in his work [4]. In the foreign mathematical literature, this method is known as Nörlund method. The history of the summation by the Voronoi functional method and its applications to trigonometric Fourier integrals, fundamental objects of this theory and review of the achieved results are contained in [1].

Let $p(t)$ be an integrable function on the real semiaxis. If $f(u)$ is integrable in $(0, \infty)$, $S(u) = \int\limits_0^u f(t)\, dt$, $P(y) = \int\limits_0^y p(t)\, dt \neq 0$, and

$$\tau(y) = \frac{1}{P(y)} \int\limits_0^y P(y-u) f(u)\, du = \frac{1}{P(y)} \int\limits_0^y p(y-u) S(u)\, du,$$

then the integral $\int\limits_0^\infty f(u)\, du$ is said to be summable by the Voronoi method, or $(W, p(t))$ is summable to I, if $\lim\limits_{y\to\infty} \tau(y) = I$.

We recall that the conditions of regularity of the method of summation are [3]

$$\int\limits_0^y |p(t)|\, dt = O(|P(y)|), \quad \text{as } y \to \infty, \tag{1}$$

Received July 3, 2018; revised November 21, 2018.
2010 *Mathematics Subject Classification.* Primary 42B08; Secondary 42B10.
Key words and phrases. Fourier integral, conjugate integral, Voronoi method, integration by parts.

and

$$\lim_{y\to\infty}\frac{1}{P(y)}\int_0^{y_0}|p(y-t)|\,dt=0 \text{ for } y_0>0. \tag{2}$$

For a function $f(t)\in L(-\infty,\infty)$, its Fourier integral (see [5]) as

$$\frac{1}{\pi}\int_0^{\infty}du\int_{-\infty}^{\infty}f(t)\cos u(x-t)\,dt, \tag{3}$$

its conjugate integral is defined as

$$\frac{1}{\pi}\int_0^{\infty}du\int_{-\infty}^{\infty}f(t)\sin u(t-x)\,dt. \tag{4}$$

We put

$$\varphi(t)=\varphi(x,t)=f(x+t)+f(x-t)-2f(x),$$
$$\psi(t)=\psi(x,t)=f(x+t)-f(x-t).$$

The following theorem holds.

Theorem 1. *a) Let $\lambda(t)$ be a function defined in $t>0$, positive and monotone non-decreasing. Suppose that $\dfrac{\lambda(t)}{t}$ is monotone non-increasing,*

$$\frac{\lambda(t)}{t}=o(1) \text{ as } t\to\infty \tag{5}$$

and

$$\int_a^t\frac{du}{\lambda(u)}=O\left(\frac{t}{\lambda(t)}\right) \text{ as } t\to\infty \tag{6}$$

for $a>0$ as fixed.

If $p(y)$ is continuously differentiable and satisfies the conditions:

$$\frac{y|p(y)|}{P(y)}=O(|P(y)|) \text{ as } y\to\infty, \tag{7}$$

$$\int_a^y\frac{t|p'(t)|}{\lambda(t)}dt=O(|P(y)|) \text{ as } y\to\infty \tag{8}$$

for $a>0$ and

$$\int_a^y\frac{P(t)}{t\lambda(t)}dt=O(|P(y)|) \text{ as } y\to\infty, \tag{9}$$

and if

$$\Phi(t) = \int_0^t |\varphi(u)|\, du = o\left(\frac{t}{\lambda(\frac{1}{t})}\right) \quad as\ t \to 0, \tag{10}$$

then the Fourier integral (3) *associated with the function* $f(t)$ *is summable by Voronoi i.e. summable* $(W, p(y))$ *to the f(x) at the point* $t = x$.

b) If we replace condition (10) *in a) by the condition*

$$\Psi(t) = \frac{1}{\pi}\int_0^t \frac{\psi(u)}{u} du = o\left(\frac{t}{\lambda(\frac{1}{t})}\right) \quad as\ t \to 0, \tag{11}$$

and if

$$g(x) = \frac{1}{\pi}\int_0^\infty \frac{\psi(t)}{t} dt = \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_\varepsilon^\infty \frac{\psi(t)}{t} dt,$$

exists and is finite, then the conjugate integral (4) *associated with the function* $f(t)$ *is summable* $(W, p(y))$ *to* $g(x)$ *at the point* $t = x$.

2. The following notations will be used in the sequel.

$$S(y) = \frac{1}{\pi}\int_0^y du \int_{-\infty}^\infty f(t)\cos u(t-x)\, dt = \frac{1}{\pi}\int_0^\infty (f(x+t) - f(x-t))\frac{\sin yt}{t} dt$$

and $S(y) - f(x) = \dfrac{1}{\pi}\int\limits_0^\infty \varphi(t)\dfrac{\sin yt}{t} dt.$

Further,

$$\tau(y) = \frac{1}{\pi P(y)}\int_0^y p(u)S(y-u)\, du$$

and

$$\begin{aligned}\tau(y) - f(x) &= \frac{1}{\pi P(y)}\int_0^y p(u)\, du \int_0^\infty \frac{\varphi(t)}{t}\sin(y-u)t\, dt = \\ &= \int_0^\infty \varphi(t)B(y,t)\, dt = \\ &= \left(\int_0^{1/y} + \int_{1/y}^{\delta} + \int_\delta^\infty\right)\varphi(t)B(y,t)\, dt = I_1 + I_2 + I_3. \end{aligned} \tag{12}$$

where $B(y,t) = \dfrac{1}{\pi P(y)} \int\limits_0^y \dfrac{p(u)\sin(y-u)t}{t}\,du$.

Next,

$$\overline{S}(y) = \frac{1}{\pi}\int\limits_0^y du \int\limits_{-\infty}^{\infty} f(t)\sin u(t-x)\,dt = \frac{1}{\pi}\int\limits_0^{\infty} \frac{\psi(t)}{t}(1-\cos yt)\,dt$$

and

$$\overline{\tau}(y) = \frac{1}{\pi P(y)}\int\limits_0^{\infty} \frac{\psi(t)}{t}\,dt \int\limits_0^y p(u)\cos(y-u)t\,dt.$$

Therefore we have

$$\begin{aligned}\overline{\tau}(y) - g(x) &= \frac{1}{\pi P(y)}\int\limits_0^{\infty} \frac{\psi(t)}{t}\,dt \int\limits_0^y p(u)\cos(y-u)t\,dt = \\ &= \int\limits_0^{\infty} \psi(t)\overline{B}(y,t)\,dt = \\ &= \left(\int\limits_0^{1/y} + \int\limits_{1/y}^{\delta} + \int\limits_{\delta}^{\infty}\right)\psi(t)\overline{B}(y,t)\,dt = \overline{I_1} + \overline{I_2} + \overline{I_3}, \qquad (13)\end{aligned}$$

where $\dfrac{1}{y}$ and $\delta > 0$ is fixed.

We set

$$\tilde{B}(y,t) = \frac{1}{\pi P(y)}\int\limits_0^y p(u)\frac{1-\cos(y-u)t}{t}\,du = \frac{1}{\pi t} + \overline{B}(y,t). \qquad (14)$$

3. We shall require six following lemmas.

Lemma 1. *If the function $p(y)$ satisfies condition (2), then there exists a fixed positive integer A such that $P(y)$ have constant sign for $y \geq A$.*

Proof. By (2), there exists an integer $A > 0$ such that

$$\left|\frac{1}{P(y)}\int\limits_0^{y_0} |p(y-t)|\,dt\right| < 1.$$

We have

$$\left|\frac{1}{P(y)}\int\limits_0^{y_0}|p(y-t)|\,dt\right| \le \frac{1}{|P(y)|}\int\limits_0^{y_0}|p(y-t)|\,dt < 1 \text{ for } y \ge A.$$

Therefore, we have

$$\left|\frac{P(y-y_0)-P(y)}{P(y)}\right| < 1 \text{ for } y \ge A,$$

and

$$0 < \frac{P(y-y_0)}{P(y)} < 2 \text{ for } y \ge A.$$

By this, $P(y)$ have constant sign for $y \ge A$. □

Lemma 2. *We have*

$$\left.\begin{matrix}|B(y,t)|\\ |\tilde{B}(y,t)|\end{matrix}\right\} = O(y) \quad as\ y \to \infty, \tag{15}$$

where $t \ge 0$.

Proof. We have

$$B(y,t)+i\tilde{B}(y,t) = \frac{1}{\pi P(y)}\int\limits_0^y \frac{p(y-u)\sin ut}{t}\,du +$$

$$+ i\frac{1}{\pi P(y)}\int\limits_0^y \frac{p(y-u)\cos ut}{t}\,du - \frac{1}{\pi t P(y)}\int\limits_0^y p(y-u)\,du =$$

$$= \frac{1}{\pi P(y)}\int\limits_0^y p(y-u)\,du\int\limits_0^u e^{-izt}\,dz.$$

Further,

$$|B(y,t)+i\tilde{B}(y,t)| \le \frac{1}{|P(y)|}\int\limits_0^y u|p(y-u)|\,du \le$$

$$\le \frac{y}{\pi P(y)}\int\limits_0^y |p(u)|\,du = O(y) \text{ as } y \to \infty$$

by (1). Therefore, $B(y,t) = O(y)$ and $\tilde{B}(y,t) = O(y)$ as $y \to \infty$. □

Lemma 3. *If we write*

$$|p(y)| = r(y), \quad R(y) = \int_0^y r(t)\,dt, \quad V(y) = \int_a^y |p'(t)|\,dt,$$

then we have

$$\left.\begin{matrix} |B(y,t)| \\ |\tilde{B}(y,t)| \end{matrix}\right\} \le \frac{1}{tP(y)}\left(R\left(\tfrac{1}{t}\right) + \tfrac{1}{t}\left\{r\left(\tfrac{1}{t}\right) - r(y) + V(y) - V\left(\tfrac{1}{t}\right)\right\}\right) \quad (16)$$

for $\dfrac{1}{y} \le t \le \delta$.

Proof. We set

$$T(y,t) = \int_0^y p(u)e^{itu}\,du = \left(\int_0^{1/t} + \int_{1/t}^y\right) p(u)e^{itu}\,du = T_1 + T_2.$$

We see that

$$|T_1| \le \int_0^{1/t} |p(u)|\,du = R\left(\frac{1}{t}\right).$$

Further, by the partial integration, we have

$$T_2 = \frac{p(y)e^{ity}}{it} - p\left(\frac{1}{t}\right)\frac{e^i}{it} - \frac{1}{it}\int_{1/t}^y p'(u)e^{itu}\,du$$

and

$$|T_2| \le \frac{1}{t}\left(|p(y)| + \left|p\left(\frac{1}{t}\right)\right| + \int_{1/t}^y |p'(u)|\,du\right) =$$

$$= \frac{1}{t}\left(|p(y)| + \left|p\left(\frac{1}{t}\right)\right| + V(y) - V\left(\frac{1}{t}\right)\right). \quad \square$$

Lemma 4. *If the function* $p(y)$ *satisfies* (9)*, then we have*

$$\int_a^y \frac{R(u)}{u\lambda(u)}\,du = O(|P(y)|) \ \text{ as } \ y \to \infty, \quad (17)$$

where $a > 0$ *is fixed.*

Proof. In view of Lemma 1, there exists $A > 0$ such that $P(y)$ have constant sign for $y > A$. Then

$$\int_a^y \frac{|P(y)|}{u\lambda(u)}\,du = \int_a^A \frac{|P(y)|}{u\lambda(u)}\,du + \int_A^y \frac{|(P(y)|}{u\lambda(u)}\,du =$$

$$= O(1) + signP(u)\int_A^y \frac{(P(y)}{u\lambda(u)}\,du$$

$$= O(1) + O(|P(y)|) = O(|P(y)|) \text{ by (9).} \quad \square$$

Lemma 5. *Let $\lambda(t)$ satisfy the conditions in the theorem. If a continuously differentiable function $p(y)$ satisfies conditions* (7) *and* (8)*, then we have*

$$\int_a^y \frac{r(t)}{\lambda(t)}\,dt = O(|P(y)|) \ \text{ as } \ y \to \infty. \tag{18}$$

Proof. By the partial integration, we have

$$\int_a^y \frac{r(t)}{\lambda(t)}\,dt = \int_a^y r(t)d\left(\int_a^t \frac{dz}{\lambda(z)}\right) = r(y)\int_a^y \frac{dz}{\lambda(z)} - \int_a^y r'(t)\,dt \int_a^t \frac{dz}{\lambda(z)}.$$

From (6), (7) and (8), we have

$$\int_a^y \frac{r(t)}{\lambda(t)}\,dt = O\left(\frac{yr(y)}{\lambda(y)}\right) + O\left(\int_a^y \frac{|r'(t)|t}{\lambda(t)}\,dt\right) =$$

$$= O(|P(y)|) + O(|P(y)|) = O(|P(y)|) \text{ as } y \to \infty. \quad \square$$

Lemma 6. *Suppose that $\lambda(t)$ satisfies the conditions, except* (6)*, in the theorem. Further, let a continuously differentiable function $p(y)$ satisfy condition* (8)*. Then we have*

$$V(y) = O(|P(y)|) \ \text{ as } \ y \to \infty.$$

Proof. Integrating by parts, we get in view (8) and (5)

$$V(y) = \int_a^y |p'(u)|\,du = \int_a^y \frac{\lambda(u)}{u}\,d\left(\int_a^u \frac{z|p'(z)|}{\lambda(z)}\right) =$$

$$= \frac{\lambda(y)}{y}\int_a^y \frac{z|p'(z)|}{\lambda(z)}\,dz - \int_a^y \left(\frac{\lambda(u)}{u}\right)' du \int_a^u \frac{z|p'(z)|}{\lambda(z)} =$$

$$= O\left(\frac{\lambda(y)}{y}|P(y)|\right) + \int_a^y \left|\left(\frac{\lambda(u)}{u}\right)'\right| du \int_a^u \frac{z|p'(z)|}{\lambda(z)}\,dz =$$

$$= o(|P(y)|) + O\left(\int_a^y \left|\left(\frac{\lambda(u)}{u}\right)'\right| \cdot |P(u)|\,du\right) =$$

$$= o(|P(y)|) + O\left(|P(u)| \int_a^y \left|\left(\frac{\lambda(u)}{u}\right)'\right| du\right) =$$

$$= o(|P(y)|) + O\left(|P(u)|\frac{\lambda(y)}{y}\right) + O(|P(y)|) =$$

$$= O(|P(y)|) \text{ as } y \to \infty. \qquad \square$$

4. Now, we shall prove the Theorem 1.

Proof of a part a). We must show that $\tau(y) - f(x) = o(1)$ as $y \to \infty$. By (12), (15), (10), we have

$$|I_1| \le O(y)\Phi\left(\frac{1}{y}\right) = O(y) \cdot o\left(\frac{\frac{1}{y}}{\lambda(y)}\right) = o(1) \text{ as } y \to \infty.$$

By (12), Riemann–Lebesgue theorem [3] and the regularity of the method of summation, we have $I_3 = o(1)$ as $y \to \infty$, where $\delta > 0$ is fixed.

Finally, we prove $I_2 = o(1)$ as $y \to \infty$, i.e.

$$\int_{1/y}^{\delta} \varphi(t)B(y,t)\,dt = o(1). \tag{19}$$

As for the notation we say that, being given an arbitrarily small $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon)$ and $A_0 = A_0(\varepsilon)$ such that the left-hand side of (19) will be $\le \varepsilon$ in absolute value for $y \ge A_0$.

By (16), we have

$$|I_2| \le \frac{1}{|P(y)|}\int_{1/y}^{\delta} \frac{|\varphi(t)|}{t} R\left(\frac{1}{t}\right) dt +$$

$$+ \frac{1}{|P(y)|}\int_{1/y}^{\delta} \frac{|\varphi(t)|}{t^2} r\left(\frac{1}{t}\right) dt + \frac{r(y)}{|P(y)|}\int_{1/y}^{\delta} \frac{|\varphi(t)|}{t^2} dt +$$

$$+\frac{1}{|P(y)|}\int\limits_{1/y}^{\delta}\frac{|\varphi(t)|}{t^2}\left\{V(y)-V\left(\frac{1}{y}\right)\right\}\,dt=J_1+J_2+J_3+J_4,\ \text{say.}$$

By the partial integration, we have

$$J_1=\frac{1}{|P(y)|}\left\{\Phi(\delta)\frac{1}{\delta}R\left(\frac{1}{\delta}\right)-y\,\Phi\left(\frac{1}{y}\right)R(y)+\right.$$
$$\left.+\int\limits_{1/y}^{\delta}\frac{\Phi(t)}{t^2}\,R\left(\frac{1}{t}\right)\,dt-\int\limits_{1/y}^{\delta}\frac{\Phi(t)}{t}\,dR\left(\frac{1}{t}\right)\right\}.$$

Here the integrated part is $o(1)$ by (10), (1) and the fact that $\lambda(t)$ is positive and monotone non-decreasing.

The second term is

$$o\left(\frac{1}{|P(y)|}\int\limits_{1/y}^{\delta}R\left(\frac{1}{t}\right)\frac{dt}{t\lambda(\frac{1}{t})}\right)=o\left(\frac{1}{|P(y)|}\int\limits_{1/\delta}^{y}R(z)\frac{dz}{z\lambda(z)}\right)=o(1)$$

by (10), (17).

The third term is

$$o\left(\frac{1}{|P(y)|}\int\limits_{1/y}^{\delta}\frac{1}{\lambda(\frac{1}{t})}dR\left(\frac{1}{t}\right)\right)=o\left(\frac{1}{|P(y)|}\int\limits_{1/\delta}^{y}\frac{R'(z)}{\lambda(z)}dz\right)=$$
$$=o\left(\frac{1}{|P(y)|}\int\limits_{1/\delta}^{y}\frac{r(z)}{\lambda(z)}dz\right)=o(1)\ \text{ by (10), (18).}$$

Therefore, we have $J_1=o(1)$ as $y\to\infty$.

Next, by the partial integration, we have

$$J_2=\frac{1}{|P(y)|}\left\{\left[\Phi(t)\frac{1}{t^2}r\left(\frac{1}{t}\right)\right]_{1/y}^{\delta}+\int\limits_{1/y}^{\delta}\frac{\Phi(t)}{t^3}r\left(\frac{1}{t}\right)\,dt-\int\limits_{1/y}^{\delta}\Phi(t)\frac{1}{t^2}dr\left(\frac{1}{t}\right)\right\}.$$

Here, the integrated part is $o(1)$ by (10), (7), and (1).

The second term is

$$o\left(\frac{1}{|P(y)|}\int\limits_{1/y}^{\delta}r\left(\frac{1}{t}\right)\frac{1}{t^2\lambda\left(\frac{1}{t}\right)}dt\right)=o\left(\frac{1}{|P(y)|}\int\limits_{1/\delta}^{y}\frac{r(z)}{\lambda(z)}dz\right)=o(1)$$

by (18).

The third term is

$$o\left(\frac{1}{|P(y)|}\int_{1/y}^{\delta}\frac{t}{\lambda\left(\frac{1}{t}\right)}\frac{1}{t^2}\left|dr\left(\frac{1}{t}\right)\right|\right)=o\left(\frac{1}{|P(y)|}\int_{1/\delta}^{y}\frac{z}{\lambda(z)}|dr(z)|\right)=$$

$$=o\left(\frac{1}{|P(y)|}\int_{1/\delta}^{y}\frac{z|r'(z)|}{\lambda(z)}dz\right)=o(1)\ \text{ by (10), (8).}$$

Therefore, we have $J_2=o(1)$.

Further, by the partial integration, we have

$$J_3=\frac{r(y)}{|P(y)|}\left\{\left[\frac{1}{t^2}\Phi(t)\right]_{1/y}^{\delta}+2\int_{1/y}^{\delta}\frac{\Phi(t)}{t^3}dt\right\}.$$

Since $\dfrac{\lambda(t)}{t}$ is positive and monotone non-increasing, $\dfrac{t}{\lambda(t)}$ is positive and monotone non-decreasing. Therefore, we have

$$J_3=o\left(\frac{y\,r(y)}{\lambda(y)|P(y)|}\right)+o\left(\frac{r(y)}{|P(y)|}\int_{1/y}^{\delta}\frac{1}{\lambda\left(\frac{1}{t}\right)}\frac{dt}{t^2}\right)=$$

$$=o\left(\frac{y\,r(y)}{\lambda(y)|P(y)|}\right)+o\left(\frac{r(y)}{|P(y)|}\int_{1/\delta}^{y}\frac{dz}{\lambda(z)}\right)=$$

$$=o\left(\frac{y\,r(y)}{\lambda(y)|P(y)|}\right)=o(1)\ \text{ by (10), (6), (7).}$$

Finally, by the partial integration, we have

$$J_4=\frac{1}{|P(y)|}\left\{\left[\Phi(t)\frac{1}{t^2}\left(V(y)-V\left(\frac{1}{t}\right)\right)\right]_{1/y}^{\delta}+\right.$$

$$\left.+\int_{1/y}^{\delta}\Phi(t)\frac{1}{t^3}\left(V(y)-V\left(\frac{1}{t}\right)\right)dt+\right.$$

$$+\int_{1/y}^{\delta}\Phi(t)\frac{1}{t^2}dV\left(\frac{1}{t}\right)\Bigg\}=J_{41}+I_{42}+J_{43}.$$

View of the fact that $\dfrac{t}{\lambda(t)}$ is positive non-decreasing, and in view (10), and (8), we have

$$J_{41}=o\left(\frac{1}{|P(y)|}\frac{\delta}{\lambda\left(\frac{1}{\delta}\right)}\frac{1}{\delta^2}\left(\int_a^y|p'(t)|\,dt-\int_a^{\delta}|p'(t)|\,dt\right)\right)=$$

$$=o\left(\frac{1}{|P(y)|}\int_{1/\delta}^{y}\frac{t|p'(t)|}{\lambda(t)}dt\right)=o(1)\ \ y\to\infty.$$

Further, integrating by parts, we get, by (6), (8), Lemma 6, and the fact that $V(u)$ is positive, monotone, and non-decreasing,

$$J_{42}=o\left(\frac{1}{|P(y)|}\int_{1/y}^{\delta}\left(V(y)-V\left(\frac{1}{t}\right)\right)\frac{dt}{t^2\lambda\left(\frac{1}{t}\right)}\right)=$$

$$=o\left(\frac{1}{|P(y)|}\int_{1/\delta}^{y}\left(V(y)-V(z)\right)\frac{dz}{\lambda(z)}\right)=$$

$$=o\left(\frac{1}{|P(y)|}\left(V(y)-V\left(\frac{1}{\delta}\right)\right)\int_a^{1/\delta}\frac{dt}{\lambda(t)}+\frac{1}{|P(y)|}\int_{1/\delta}^{y}\frac{z|p'(z)|}{\lambda(z)}dz\right)=$$

$$=o(1)\ \text{ by (6), (8), (10) and Lemma 6.}$$

Finally by (10) we get

$$J_{43}=o\left(\frac{1}{|P(y)|}\int_{1/y}^{\delta}\frac{t}{\lambda\left(\frac{1}{t}\right)}\cdot\frac{1}{t^2}dV\left(\frac{1}{t}\right)\right)=$$

$$=o\left(\frac{1}{|P(y)|}\int_{1/\delta}^{y}\frac{z|p'(z)|}{\lambda(z)}dz\right)=o(1)\ \text{ by (8).}$$

Therefore, we have $J_4 = o(1)$ and $I_2 = o(1)$. We get $I_1 = o(1)$, $I_2 = o(1)$, $I_3 = o(1)$. By (12), we have that $\tau(y) - f(x) = o(1)$ as $y \to \infty$ i.e. $\lim\limits_{y\to\infty} \tau(y) = f(x)$.

Proof of a part b). $\overline{I}_2 = o(1)$ and $\overline{I}_3 = o(1)$ will be proved by the same method as the relations I_2 and I_3.

Let us evaluate $\overline{I}_1$. By (13) and (14), we have

$$\overline{I}_1 = \lim_{\varepsilon\to 0} \int_{\varepsilon}^{1/y} \psi(t)\tilde{B}(y,t)\,dt + \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{\varepsilon}^{1/y} \frac{\psi(t)}{t}\,dt.$$

The first expression on the right is

$$O\left(y\,\Phi\left(\frac{1}{y}\right)\right) = o(1) \text{ as } y \to \infty, \text{ by (15) and (11).}$$

The second expression on the right is $o(1)$, as $y \to \infty$, since $g(x)$ exists and is finite.

5. The derived Fourier integral of a function $f(t) \in L(-\infty, \infty)$ is

$$\frac{1}{\pi}\int_0^{\infty} u\,du \int_{-\infty}^{\infty} f(t)\sin u(t-x)\,dt. \tag{20}$$

Then its conjugate derived Fourier integral is given by

$$-\frac{1}{\pi}\int_0^{\infty} u\,du \int_{-\infty}^{\infty} f(t)\cos u(t-x)\,dt. \tag{21}$$

From Theorem 1, we get the following theorem [2].

Theorem 2. *Let $f(t)$ be an absolutely continuous function over every finite interval and which is of bounded variation over $(-\infty, \infty)$. Let $p(t)$ and $\lambda(t)$ be defined as in Theorem 1.*

a) If

$$\int_0^t |d_u\{f(x+u) - f(x-u) - 2uf'(x)\}| = o\left(\frac{t}{\lambda\left(\frac{1}{t}\right)}\right) \text{ as } t \to 0,$$

then the derived Fourier integral (20) *is summable $(W, p(y))$ to $f'(x)$ at the point $t = x$.*

b) If

$$\int_0^t |d_u\{f(x+u) + f(x-u)\}| = o\left(\frac{t}{\lambda\left(\frac{1}{t}\right)}\right) \text{ as } t \to 0,$$

then the conjugate derived Fourier integral (21) *is summable* $(W, p(y))$ *to*

$$\frac{1}{\pi}\int_0^\infty \frac{f(x+t)+f(x-t)-2f(x)}{t}\,dt$$

at every point, at which the above integral exists.

6. In the case $\lambda(t) = 1$, the sufficient conditions are stated. They are imposed on the function $p(y)$ defining the functional Voronoi method and on the point of the function $f(t)$, under which integrals (3), (4), (20), (21) are summable $(W, p(y))$.

References

1. L. Boytsun, *Functional Voronoi method*, Voronoi's Impact on Modern Science, Proc. of Inst. Math. of NASU, vol. 21, Institute of Mathematics, Kyiv, 1998, pp. 199–206.
2. L. G. Boytsun, *On summation of Fourier trigonometrical integrals by the Voronoi method*, Proc. of Higher Education, Math. **10** (1971), 16–23.
3. K. Knopp and B. Vanderberg, *Functional Nörlund methods.* I, Rend. Circ. Mat. Palermo **4** (1955), no. 1, 5–32.
4. G. Voronoi, *Extension of notion of the limit of the sum of terms of an infinite series*, Works in 3 Volumes, Vol. 3, Kyiv, 1952, pp. 9–10.
5. E. C. Titchmarsh, *Introduction to the theory of Fourier integrals*, Chelsea Publ., New York, 1986.

Department of Mathematical analysis and Theory of Functions, Oles Honchar Dnipro National University, 72 Gagarin Av., Dnipro, 41010, Ukraine
Email: lgboytsun@gmail.com

Department of Mathematical analysis and Theory of Functions, Oles Honchar Dnipro National University, 72 Gagarin Av., Dnipro, 41010, Ukraine
Email: t.rybnikova@gmail.com

Voronoï's Impact on Modern Science, Book 6, Volume 1
Proceedings of the Sixth International Conference on Analytic Number Theory and Spatial Tessellations

LIMIT BEHAVIOR OF THE EXPECTED ESSCHER TRANSFORMATION

Vitaliy Drozdenko (*Ukraine*)

Abstract. We present two limit theorems describing the convergence of an expected Esscher transform to the essential supremum of a variable under a transformation, when the transformation parameter tends to plus infinity, as well as the convergence of the same object to the essential infimum of a variable under a transformation when the transformation parameter tends to minus infinity.

Introduction

Let us consider an arbitrary random variable X with distribution function $F_X(x)$. The *Esscher transform* of the random variable X is defined as

$$\mathrm{ET}[X;\,\alpha] := \frac{Xe^{\alpha X}}{\mathsf{E}[e^{\alpha X}]},$$

where α is a real parameter. The *expected Esscher transform* of the random variable X is defined as the expectation of the corresponding Esscher transform,

$$\mathrm{EET}[X;\,\alpha] := \mathsf{E}\left[\mathrm{ET}[X;\,\alpha]\right] = \frac{\mathsf{E}[Xe^{\alpha X}]}{\mathsf{E}[e^{\alpha X}]},$$

where α is a real parameter.

The Esscher transform was introduced in 1932 by a Swedish mathematician, Fredrik Esscher. The description of known results related to this theme can be found in the review by Yang (2004).

To analyze the limit behavior of the expected Esscher transform, we need the following two definitions: *essential infimum* of the random variable X, namely,

$$\operatorname{ess\,inf}[X] := \inf\{\delta :\ F_X(\delta) > 0\};$$

Received November 8, 2018; revised November 20, 2018.

2010 *Mathematics Subject Classification.* Primary 60F17, 60F99; Secondary 91B30.

Key words and phrases. Esscher transformation, Esscher premium, limit behavior, convergence, essential supremum, essential infimum, insurance premium.

and *essential supremum* of the random variable X, namely,

$$\text{ess sup}[X] := \sup\{\delta : \ F_X(\delta) < 1\}.$$

To make our notations a bit more compact, we introduce the abbreviations:

$$\underline{b} := \text{ess inf}[X] \quad \text{and} \quad \overline{b} := \text{ess sup}[X].$$

1. Limit Theorems

The expected Esscher transform possesses the following limit behavior, when the transformation parameter grows to plus infinity.

Theorem 1. *For any random variable X, the limit relation*

$$\lim_{\alpha\to+\infty} \text{EET}[X;\,\alpha] = \text{ess sup}[X] \tag{1}$$

holds.

Proof. For the convenience of computations, by proving Theorem 1, we consider separately the following several cases:

Case 1: $\text{ess inf}[X] = \text{ess sup}[X] < +\infty$;
Case 2: $\text{ess inf}[X] < \text{ess sup}[X] \le 0$;
Case 3: $\text{ess inf}[X] < 0 < \text{ess sup}[X] < +\infty$;
Case 4: $0 \le \text{ess inf}[X] < \text{ess sup}[X] < +\infty$;
Case 5: $0 \le \text{ess inf}[X] < \text{ess sup}[X] = +\infty$;
Case 6: $\text{ess inf}[X] < 0$, and $\text{ess sup}[X] = +\infty$.

First, let us show that the limit expected Esscher transform for any random variable X will not exceed the essential supremum of the random variable X. Indeed,

$$\begin{aligned}\lim_{\alpha\to+\infty} \text{EET}[X;\,\alpha] &= \lim_{\alpha\to+\infty} \frac{\mathsf{E}[Xe^{\alpha X}]}{\mathsf{E}[e^{\alpha X}]} \le \lim_{\alpha\to+\infty} \frac{\mathsf{E}[\text{ess sup}[X]e^{\alpha X}]}{\mathsf{E}[e^{\alpha X}]} \\ &\le \lim_{\alpha\to+\infty} \frac{\text{ess sup}[X]\mathsf{E}[e^{\alpha X}]}{\mathsf{E}[e^{\alpha X}]} = \text{ess sup}[X]. \end{aligned} \tag{2}$$

Case 1 corresponds to the situation where the random variable X is equal to a constant with probability 1, i.e., $\mathsf{P}\{X = C\} = 1$ for the constant C such that $C = \text{ess inf}[X] = \text{ess sup}[X]$, and, in this case,

$$\lim_{\alpha\to+\infty} \text{EET}_{\textbf{case 1}}[X;\,\alpha] = \lim_{\alpha\to+\infty} \frac{Ce^{\alpha C}}{e^{\alpha C}} = C = \text{ess sup}[X].$$

Hence, the limit relation (1) holds in the first considered case.

Since we analyze the limit behavior of stochastic functionals, when the parameter α tends to plus infinity, without loss of generality, the parameter α can be treated as strictly positive.

In the second, third, and fourth considered cases for any ε satisfying the inequality

$$0 < \varepsilon < \operatorname{ess\,sup}[X] - \operatorname{ess\,inf}[X], \tag{3}$$

the integral limit relation

$$\int_{\underline{b}}^{\overline{b}-\varepsilon} e^{\alpha x} dF_X(x) = o\left(\int_{\overline{b}-\varepsilon}^{\overline{b}} e^{\alpha x} dF_X(x)\right) \quad \text{as} \quad \alpha \to +\infty \tag{4}$$

holds. To prove relation (4), we need the inequalities

$$\int_{\underline{b}}^{\overline{b}-\varepsilon} e^{\alpha x} dF_X(x) < e^{\alpha(\overline{b}-\varepsilon)} \mathsf{P}\{X \le \overline{b} - \varepsilon\}, \tag{5}$$

and

$$\int_{\overline{b}-\varepsilon}^{\overline{b}} e^{\alpha x} dF_X(x) \;\ge\; \int_{\overline{b}-\varepsilon/2}^{\overline{b}} e^{\alpha x} dF_X(x) \;>\; e^{\alpha\left(\overline{b}-\varepsilon/2\right)} \mathsf{P}\left\{X \ge \overline{b} - \varepsilon/2\right\} > 0. \tag{6}$$

Using inequalities (5) and (6), we get

$$\begin{aligned} 0 < \frac{\int_{\underline{b}}^{\overline{b}-\varepsilon} e^{\alpha x} dF_X(x)}{\int_{\overline{b}-\varepsilon}^{\overline{b}} e^{\alpha x} dF_X(x)} &< \frac{\mathsf{P}\{X \le \overline{b} - \varepsilon\} e^{\alpha(\overline{b}-\varepsilon)}}{e^{\alpha\left(\overline{b}-\varepsilon/2\right)} \mathsf{P}\left\{X \ge \overline{b} - \varepsilon/2\right\}} \\ &= \frac{\mathsf{P}\{X \le \overline{b} - \varepsilon\}}{\mathsf{P}\left\{X \ge \overline{b} - \varepsilon/2\right\}} e^{-\frac{\alpha\varepsilon}{2}} = \text{constant} \cdot e^{-\frac{\alpha\varepsilon}{2}} \to 0 \quad \text{as} \quad \alpha \to +\infty. \end{aligned}$$

Hence, the asymptotic relation (4) holds, indeed.

In the second and fourth cases for any ε satisfying (3), the expected Esscher transform can be represented as follows:

$$\mathrm{EET}_{\textbf{cases 2 and 4}}[X;\, \alpha] = \frac{\int_{\underline{b}}^{\overline{b}-\varepsilon} x e^{\alpha x} dF_X(x) + \int_{\overline{b}-\varepsilon}^{\overline{b}} x e^{\alpha x} dF_X(x)}{\int_{\underline{b}}^{\overline{b}-\varepsilon} e^{\alpha x} dF_X(x) + \int_{\overline{b}-\varepsilon}^{\overline{b}} e^{\alpha x} dF_X(x)}. \tag{7}$$

Let us show that, in the second case, the limit relation

$$\int_{\underline{b}}^{\overline{b}-\varepsilon} x e^{\alpha x} dF_X(x) = o\left(\int_{\overline{b}-\varepsilon}^{\overline{b}} e^{\alpha x} dF_X(x)\right) \quad \text{as} \quad \alpha \to +\infty, \tag{8}$$

holds. In other words, the first summand in the numerator of representation (7) will be negligible with respect to the second summand in the denominator of representation (7), when the parameter α tends to plus infinity.

Indeed, for the parameter $\alpha > 0$, the function $xe^{\alpha x}$ decreases for $x \in (-\infty, -1/\alpha)$, increases for $x \in (-1/\alpha, +\infty)$, and is negative for

all $x \le \overline{b}-\varepsilon < 0$. For $-1/\alpha \ge \overline{b}-\varepsilon$, i.e., for $\alpha \ge 1/(\varepsilon-\overline{b})$, the first summand in the numerator of representation (7) can be majorized as follows:

$$\left|\int_{\underline{b}}^{\overline{b}-\varepsilon} xe^{\alpha x}dF_X(x)\right| \;<\; \left|(\overline{b}-\varepsilon)e^{\alpha(\overline{b}-\varepsilon)}\mathsf{P}\left\{X \le \overline{b}-\varepsilon\right\}\right|$$
$$= \;(\varepsilon-\overline{b})e^{\alpha(\overline{b}-\varepsilon)}\mathsf{P}\left\{X \le \overline{b}-\varepsilon\right\}. \qquad (9)$$

Combining inequality (9) with inequality (6), we get

$$0 \;<\; \frac{\left|\int_{\underline{b}}^{\overline{b}-\varepsilon} xe^{\alpha x}dF_X(x)\right|}{\int_{\overline{b}-\varepsilon}^{\overline{b}} e^{\alpha x}dF_X(x)} \;<\; \frac{(\varepsilon-\overline{b})e^{\alpha(\overline{b}-\varepsilon)}\mathsf{P}\left\{X \le \overline{b}-\varepsilon\right\}}{e^{\alpha\left(\overline{b}-\varepsilon/2\right)}\mathsf{P}\left\{X \ge \overline{b}-\varepsilon/2\right\}}$$
$$= \;\text{constant}\cdot e^{-\frac{\alpha\varepsilon}{2}} \;\to\; 0 \quad \text{as} \quad \alpha \to +\infty.$$

Hence, the asymptotic relation (8) holds.

Dividing the numerator and denominator of representation (7) by the integral $\int_{\overline{b}-\varepsilon}^{\overline{b}} e^{\alpha x}dF_X(x)$ and using the limit relations (4) and (8), we obtain

$$\mathrm{EET}_{\textbf{case 2}}[X;\,\alpha] \;=$$
$$= \;\frac{\left(\int_{\underline{b}}^{\overline{b}-\varepsilon} xe^{\alpha x}dF_X(x) + \int_{\overline{b}-\varepsilon}^{\overline{b}} xe^{\alpha x}dF_X(x)\right)\Big/ \int_{\overline{b}-\varepsilon}^{\overline{b}} e^{\alpha x}dF_X(x)}{\int_{\underline{b}}^{\overline{b}-\varepsilon} e^{\alpha x}dF_X(x)\Big/ \int_{\overline{b}-\varepsilon}^{\overline{b}} e^{\alpha x}dF_X(x) + 1}$$
$$\sim \int_{\overline{b}-\varepsilon}^{\overline{b}} xe^{\alpha x}dF_X(x)\Big/ \int_{\overline{b}-\varepsilon}^{\overline{b}} e^{\alpha x}dF_X(x) \quad \text{as} \quad \alpha \to +\infty$$
$$\ge (\overline{b}-\varepsilon)\int_{\overline{b}-\varepsilon}^{\overline{b}} e^{\alpha x}dF_X(x)\Big/ \int_{\overline{b}-\varepsilon}^{\overline{b}} e^{\alpha x}dF_X(x) \;=\; \overline{b}-\varepsilon.$$

Combining inequalities (2) and (10), we conclude that

$$\lim_{\alpha\to+\infty} \mathrm{EET}_{\textbf{case 2}}[X;\,\alpha] \in \bigcap_{0<\varepsilon<\overline{b}-\underline{b}} [\overline{b}-\varepsilon, \overline{b}] \equiv \left\{\overline{b}\right\}.$$

Hence, the limit relation (1) holds in the second case.

In the fourth case, the first summand in the numerator of representation (7) is non-negative. This allows us to write the following lower bound for the expected Esscher transform:

$$\mathrm{EET}_{\textbf{case 4}}[X;\,\alpha] \;\ge\; \frac{\int_{\overline{b}-\varepsilon}^{\overline{b}} xe^{\alpha x}dF_X(x)}{\int_{\underline{b}}^{\overline{b}-\varepsilon} e^{\alpha x}dF_X(x) + \int_{\overline{b}-\varepsilon}^{\overline{b}} e^{\alpha x}dF_X(x)}$$
$$=: \mathrm{EET}^{\textbf{estimate}}_{\textbf{case 4}}[X;\,\alpha].$$

Dividing the numerator and denominator of estimate $\mathrm{EET}^{\mathbf{estimate}}_{\mathbf{case\ 4}}[X;\,\alpha]$ by the integral $\int_{\overline{b}-\varepsilon}^{\overline{b}} e^{\alpha x} dF_X(x)$, switching to the limit, as α grows to infinity, and using the asymptotic relation (4), we obtain

$$\begin{aligned}
\mathrm{EET}^{\mathbf{estimate}}_{\mathbf{case\ 4}}[X;\,\alpha] \;&=\\
&= \frac{\int_{\overline{b}-\varepsilon}^{\overline{b}} x e^{\alpha x} dF_X(x) \Big/ \int_{\overline{b}-\varepsilon}^{\overline{b}} e^{\alpha x} dF_X(x)}{\int_{\underline{b}}^{\overline{b}-\varepsilon} e^{\alpha x} dF_X(x) \Big/ \int_{\overline{b}-\varepsilon}^{\overline{b}} e^{\alpha x} dF_X(x) + 1}\\
&\sim \int_{\overline{b}-\varepsilon}^{\overline{b}} x e^{\alpha x} dF_X(x) \Big/ \int_{\overline{b}-\varepsilon}^{\overline{b}} e^{\alpha x} dF_X(x) \quad \text{as} \quad \alpha \to +\infty\\
&\geq (\overline{b}-\varepsilon) \int_{\overline{b}-\varepsilon}^{\overline{b}} e^{\alpha x} dF_X(x) \Big/ \int_{\overline{b}-\varepsilon}^{\overline{b}} e^{\alpha x} dF_X(x) \;=\; \overline{b}-\varepsilon.
\end{aligned}$$

Hence, in the fourth case,

$$\lim_{\alpha\to+\infty} \mathrm{EET}_{\mathbf{case\ 4}}[X;\,\alpha] \geq \lim_{\alpha\to+\infty} \mathrm{EET}^{\mathbf{estimate}}_{\mathbf{case\ 4}}[X;\,\alpha] \geq \overline{b}-\varepsilon. \tag{10}$$

Combining inequality (10) with inequalities (2) and (3), we conclude that

$$\lim_{\alpha\to+\infty} \mathrm{EET}_{\mathbf{case\ 4}}[X;\,\alpha] \in \bigcap_{0<\varepsilon<\overline{b}-\underline{b}} [\overline{b}-\varepsilon, \overline{b}] \equiv \left\{\overline{b}\right\}.$$

Thus, the asymptotic relation (1) holds also in the fourth case.

In the third case for any ε satisfying the inequalities

$$0 < \varepsilon < \operatorname{ess\,sup}[X], \tag{11}$$

the expected Esscher transform can be represented as

$$\begin{aligned}
\mathrm{EET}_{\mathbf{case\ 3}}[X;\,\alpha] \;&=\\
&= \frac{\int_{\underline{b}}^{0} x e^{\alpha x} dF_X(x) + \int_{0}^{\overline{b}-\varepsilon} x e^{\alpha x} dF_X(x) + \int_{\overline{b}-\varepsilon}^{\overline{b}} x e^{\alpha x} dF_X(x)}{\int_{\underline{b}}^{\overline{b}-\varepsilon} e^{\alpha x} dF_X(x) + \int_{\overline{b}-\varepsilon}^{\overline{b}} e^{\alpha x} dF_X(x)}\\
&\geq \frac{\int_{\underline{b}}^{0} x e^{\alpha x} dF_X(x) + \int_{\overline{b}-\varepsilon}^{\overline{b}} x e^{\alpha x} dF_X(x)}{\int_{\underline{b}}^{\overline{b}-\varepsilon} e^{\alpha x} dF_X(x) + \int_{\overline{b}-\varepsilon}^{\overline{b}} e^{\alpha x} dF_X(x)} \;=:\; \mathrm{EET}^{\mathbf{estimate}}_{\mathbf{case\ 3}}[X;\,\alpha],
\end{aligned}$$

where the last inequality holds due to the non-negativity of

$$\int_{0}^{\overline{b}-\varepsilon} x e^{\alpha x} dF_X(x).$$

Let us show that, in the third case for any ε satisfying inequality (11), the limit relation

$$\int_{\underline{b}}^{0} xe^{\alpha x} dF_X(x) = o\left(\int_{\overline{b}-\varepsilon}^{\overline{b}} e^{\alpha x} dF_X(x) \right) \quad \text{as} \quad \alpha \to +\infty \tag{12}$$

holds. In other words, the first summand in the numerator of representation $\mathrm{EET}^{\mathbf{estimate}}_{\mathbf{case\ 3}}[X;\,\alpha]$ is negligible with respect to the second summand in the denominator of representation $\mathrm{EET}^{\mathbf{estimate}}_{\mathbf{case\ 3}}[X;\,\alpha]$, as α tends to plus infinity.

As was already mentioned, the function $xe^{\alpha x}$ for positive values of the parameter α takes negative values on the negative half-line and achieves its minimum at the point $x = -1/\alpha$. Hence,

$$\begin{aligned} \left| \int_{\underline{b}}^{0} xe^{\alpha x} dF_X(x) \right| &\leq \left| (-1/\alpha) e^{\alpha(-1/\alpha)} \mathsf{P}\left\{X \leq 0\right\} \right| \\ &= (1/\alpha) e^{-1} \mathsf{P}\left\{X \leq 0\right\}. \end{aligned} \tag{13}$$

Note that, in the third case, ϵ satisfying inequalities (11) also satisfies inequalities (3). Combining inequality (13) with inequality (6), we get

$$\begin{aligned} 0 \;<\; \frac{\left| \int_{\underline{b}}^{0} xe^{\alpha x} dF_X(x) \right|}{\int_{\overline{b}-\varepsilon}^{\overline{b}} e^{\alpha x} dF_X(x)} \;<\; & \frac{(1/\alpha) e^{-1} \mathsf{P}\left\{X \leq 0\right\}}{e^{\alpha\left(\overline{b}-\varepsilon/2\right)} \mathsf{P}\left\{X \geq \overline{b} - \varepsilon/2\right\}} \;= \\ &= \frac{\text{constant}}{\alpha} \cdot e^{-\alpha\left(\overline{b}-\varepsilon/2\right)} \;\to\; 0 \quad \text{as} \quad \alpha \to +\infty. \end{aligned}$$

Thus, in the third case, the limit relation (12) holds.

Dividing the numerator and denominator of $\mathrm{EET}^{\mathbf{estimate}}_{\mathbf{case\ 3}}[X;\,\alpha]$ by the integral $\int_{\overline{b}-\varepsilon}^{\overline{b}} e^{\alpha x} dF_X(x)$, switching to the limit as α tends to plus infinity, and using the limit relations (4) and (12), we obtain

$$\begin{aligned} &\mathrm{EET}^{\mathbf{estimate}}_{\mathbf{case\ 3}}[X;\,\alpha] \;= \\ &\quad= \frac{\left(\int_{\underline{b}}^{0} xe^{\alpha x} dF_X(x) + \int_{\overline{b}-\varepsilon}^{\overline{b}} xe^{\alpha x} dF_X(x) \right) \Big/ \int_{\overline{b}-\varepsilon}^{\overline{b}} e^{\alpha x} dF_X(x)}{\int_{\underline{b}}^{\overline{b}-\varepsilon} e^{\alpha x} dF_X(x) \Big/ \int_{\overline{b}-\varepsilon}^{\overline{b}} e^{\alpha x} dF_X(x) + 1} \\ &\quad\sim \int_{\overline{b}-\varepsilon}^{\overline{b}} xe^{\alpha x} dF_X(x) \Big/ \int_{\overline{b}-\varepsilon}^{\overline{b}} e^{\alpha x} dF_X(x) \quad \text{as} \quad \alpha \to +\infty \\ &\quad\geq (\overline{b} - \varepsilon) \int_{\overline{b}-\varepsilon}^{\overline{b}} e^{\alpha x} dF_X(x) \Big/ \int_{\overline{b}-\varepsilon}^{\overline{b}} e^{\alpha x} dF_X(x) \;=\; \overline{b} - \varepsilon. \end{aligned} \tag{14}$$

Inequality (14) for any ε satisfying (11) yields

$$\lim_{\alpha\to+\infty} \mathrm{EET}_{\textbf{case 3}}[X;\,\alpha] \geq \lim_{\alpha\to+\infty} \mathrm{EET}^{\textbf{estimate}}_{\textbf{case 3}}[X;\,\alpha] \geq \overline{b} - \varepsilon. \tag{15}$$

Combining inequalities (2), (15), and (11), we conclude that

$$\lim_{\alpha\to+\infty} \mathrm{EET}_{\textbf{case 3}}[X;\,\alpha] \in \bigcap_{0<\varepsilon<\overline{b}} [\overline{b}-\varepsilon, \overline{b}] \equiv \left\{\overline{b}\right\}.$$

Hence, the asymptotic relation (1) holds also in the third case.

In the fifth and sixth cases for any real constant c satisfying the inequality

$$\max\{0,\, \mathrm{ess\ inf}[X]\} < c < +\infty, \tag{16}$$

the integral limit relation

$$\int_{\underline{b}}^{c} e^{\alpha x} dF_X(x) = o\left(\int_{c}^{+\infty} e^{\alpha x} dF_X(x)\right) \quad \text{as} \quad \alpha \to +\infty \tag{17}$$

holds. Indeed, due to the monotonicity of the function $e^{\alpha x}$, the inequality

$$\int_{\underline{b}}^{c} e^{\alpha x} dF_X(x) < e^{\alpha c}\mathsf{P}\{X \leq c\} \tag{18}$$

holds. On the other hand,

$$\int_{c}^{+\infty} e^{\alpha x} dF_X(x) \;\geq\; \int_{2c}^{+\infty} e^{\alpha x} dF_X(x) \;>\; e^{2\alpha c}\mathsf{P}\{X \geq 2c\} \;>\; 0. \tag{19}$$

Using inequalities (18) and (19), we can write

$$\begin{aligned} 0 \;\leq\; \frac{\int_{\underline{b}}^{c} e^{\alpha x} dF_X(x)}{\int_{c}^{+\infty} e^{\alpha x} dF_X(x)} \;&<\; \frac{e^{\alpha c}\mathsf{P}\{X \leq c\}}{e^{2\alpha c}\mathsf{P}\{X \geq 2c\}} \\ &=\; \text{constant} \cdot e^{-\alpha c} \to 0 \quad \text{as} \quad \alpha \to +\infty. \end{aligned}$$

Hence, the asymptotic relation (17) holds.

In the fifth case for any $c > \underline{b}$, the expected Esscher transform can be represented as

$$\begin{aligned} \mathrm{EET}_{\textbf{case 5}}[X;\,\alpha] \;&=\; \frac{\int_{\underline{b}}^{c} x e^{\alpha x} dF_X(x) + \int_{c}^{+\infty} x e^{\alpha x} dF_X(x)}{\int_{\underline{b}}^{c} e^{\alpha x} dF_X(x) + \int_{c}^{+\infty} e^{\alpha x} dF_X(x)} \\ &\geq\; \frac{\int_{c}^{+\infty} x e^{\alpha x} dF_X(x)}{\int_{\underline{b}}^{c} e^{\alpha x} dF_X(x) + \int_{c}^{+\infty} e^{\alpha x} dF_X(x)} \;=:\; \mathrm{EET}^{\textbf{estimate}}_{\textbf{case 5}}[X;\,\alpha], \end{aligned}$$

where the last inequality holds due to the non-negativity of

$$\int_{\underline{b}}^{c} x e^{\alpha x} dF_X(x).$$

Dividing the numerator and denominator of $\mathrm{EET}^{\mathbf{estimate}}_{\mathbf{case\ 5}}[X;\,\alpha]$ by the integral $\int_c^{+\infty} e^{\alpha x} dF_X(x)$, switching to the limit as α tends to plus infinity, and using the limit relation (17), we get

$$\begin{aligned}\mathrm{EET}^{\mathbf{estimate}}_{\mathbf{case\ 5}}[X;\,\alpha] \;&=\; \frac{\int_c^{+\infty} x e^{\alpha x} dF_X(x) \Big/ \int_c^{+\infty} e^{\alpha x} dF_X(x)}{\int_{\underline{b}}^{c} e^{\alpha x} dF_X(x) \Big/ \int_c^{+\infty} e^{\alpha x} dF_X(x) + 1} \\ &\sim\; \int_c^{+\infty} x e^{\alpha x} dF_X(x) \Big/ \int_c^{+\infty} e^{\alpha x} dF_X(x) \quad \text{as} \quad \alpha \to +\infty \\ &\geq\; c \int_c^{+\infty} e^{\alpha x} dF_X(x) \Big/ \int_c^{+\infty} e^{\alpha x} dF_X(x) \;=\; c. \end{aligned} \tag{20}$$

Inequality (20) implies that, for any $c > \underline{b}$,

$$\lim_{\alpha\to+\infty} \mathrm{EET}_{\mathbf{case\ 5}}[X;\,\alpha] \geq \lim_{\alpha\to+\infty} \mathrm{EET}^{\mathbf{estimate}}_{\mathbf{case\ 5}}[X;\,\alpha] \geq c$$

and

$$\lim_{\alpha\to+\infty} \mathrm{EET}_{\mathbf{case\ 5}}[X;\,\alpha] \in \bigcap_{c>\underline{b}} [c, +\infty] \equiv \{+\infty\}.$$

Hence, the asymptotic relation (1) holds also in the fifth case.

In the sixth case for any real c satisfying the inequality

$$0 < c < +\infty \tag{21}$$

[we note that, in the sixth case, c satisfying inequality (21) satisfies inequality (16)], the expected Esscher transform can be represented as

$$\begin{aligned}\mathrm{EET}_{\mathbf{case\ 6}}[X;\,\alpha] &= \frac{\int_{\underline{b}}^{0} x e^{\alpha x} dF_X(x) + \int_0^c x e^{\alpha x} dF_X(x) + \int_c^{+\infty} x e^{\alpha x} dF_X(x)}{\int_{\underline{b}}^{c} e^{\alpha x} dF_X(x) + \int_c^{+\infty} e^{\alpha x} dF_X(x)} \\ &\geq\; \frac{\int_{\underline{b}}^{0} x e^{\alpha x} dF_X(x) + \int_c^{+\infty} x e^{\alpha x} dF_X(x)}{\int_{\underline{b}}^{c} e^{\alpha x} dF_X(x) + \int_c^{+\infty} e^{\alpha x} dF_X(x)} \;=:\; \mathrm{EET}^{\mathbf{estimate}}_{\mathbf{case\ 6}}[X;\,\alpha], \end{aligned}$$

where the last inequality is valid due to the non-negativity of

$$\int_0^c x e^{\alpha x} dF_X(x).$$

Let us show that, in the sixth case, the limit relation

$$\int_{\underline{b}}^{0} x e^{\alpha x} dF_X(x) = o\left(\int_c^{+\infty} e^{\alpha x} dF_X(x) \right) \quad \text{as} \quad \alpha \to +\infty \tag{22}$$

holds. In other words, the first summand in the numerator of representation $\mathrm{EET}^{\mathbf{estimate}}_{\mathbf{case\ 6}}[X;\,\alpha]$ is negligible with respect to the second summand

in the denominator of representation $\mathrm{EET}^{\mathbf{estimate}}_{\mathbf{case\ 6}}[X;\,\alpha]$ as α tends to plus infinity.

Indeed, combining inequalities (13) and (19), we obtain

$$0 \;\le\; \frac{\left|\int_{\underline{b}}^{0} xe^{\alpha x}dF_X(x)\right|}{\int_c^{+\infty} e^{\alpha x}dF_X(x)} \;<\; \frac{(1/\alpha)e^{-1}\mathsf{P}\left\{X\le 0\right\}}{e^{2\alpha c}\mathsf{P}\{X\ge 2c\}} = \frac{\text{constant}}{\alpha}\cdot e^{-\alpha c}\to 0 \quad \text{as} \quad \alpha\to+\infty.$$

Hence, the asymptotic relation (22) holds.

Dividing the numerator and denominator of $\mathrm{EET}^{\mathbf{estimate}}_{\mathbf{case\ 6}}[X;\,\alpha]$ by the integral $\int_c^{+\infty} e^{\alpha x}dF_X(x)$, switching to the limit as α tends to plus infinity, and using the limit relations (17) and (22), we have

$$\begin{aligned}\mathrm{EET}^{\mathbf{estimate}}_{\mathbf{case\ 6}}[X;\,\alpha] \;&=\; \frac{\left(\int_{\underline{b}}^{0} xe^{\alpha x}dF_X(x)+\int_c^{+\infty} xe^{\alpha x}dF_X(x)\right)\Big/\int_c^{+\infty} e^{\alpha x}dF_X(x)}{\int_{\underline{b}}^{c} e^{\alpha x}dF_X(x)\Big/\int_c^{+\infty} e^{\alpha x}dF_X(x)+1}\\ &\sim\; \int_c^{+\infty} xe^{\alpha x}dF_X(x)\Big/\int_c^{+\infty} e^{\alpha x}dF_X(x) \quad \text{as} \quad \alpha\to+\infty\\ &\ge c\int_c^{+\infty} e^{\alpha x}dF_X(x)\Big/\int_c^{+\infty} e^{\alpha x}dF_X(x) \;=\; c. \qquad (23)\end{aligned}$$

Inequality (23) for any c satisfying (21) yields

$$\lim_{\alpha\to+\infty}\mathrm{EET}_{\mathbf{case\ 6}}[X;\,\alpha]\ge\lim_{\alpha\to+\infty}\mathrm{EET}^{\mathbf{estimate}}_{\mathbf{case\ 6}}[X;\,\alpha]\ge c$$

and

$$\lim_{\alpha\to+\infty}\mathrm{EET}_{\mathbf{case\ 6}}[X;\,\alpha]\in\bigcap_{c>0}[c,+\infty]\equiv\{+\infty\}.$$

Hence, the asymptotic relation (1) holds also in the sixth case.

This completes the proof of the theorem. □

The expected Esscher transform possesses the following limit behavior, as the transformation parameter tends to minus infinity.

Theorem 2. *For any random variable X, the limit relation*

$$\lim_{\alpha\to-\infty}\mathrm{EET}[X;\,\alpha]=\operatorname{ess\,inf}[X] \qquad (24)$$

holds.

Proof. Using the statement of Theorem 1, we obtain

$$\begin{aligned}\lim_{\alpha\to-\infty}\mathrm{EET}[X;\,\alpha] \;&=\; \lim_{\alpha\to-\infty}\frac{\mathsf{E}[Xe^{\alpha X}]}{\mathsf{E}[e^{\alpha X}]} \;=\; -\lim_{\alpha\to-\infty}\frac{\mathsf{E}[-Xe^{-\alpha(-X)}]}{\mathsf{E}[e^{-\alpha(-X)}]}\\ &=\; -\lim_{\beta\to+\infty}\frac{\mathsf{E}[-Xe^{\beta(-X)}]}{\mathsf{E}[e^{\beta(-X)}]} \;=\; -\lim_{\beta\to+\infty}\mathrm{EET}[-X;\,\beta]\\ &=\; -\text{ess sup}[-X] \;=\; \text{ess inf}[X].\end{aligned}$$

Hence, Theorem 2 can be considered as a corollary to Theorem 1. □

Remark 1. Note that, for any random variable X and for any $\alpha\in\mathbb{R}$, the inequalities $\text{ess inf}[X]\le\mathrm{EET}[X;\,\alpha]\le\text{ess sup}[X]$ hold. This means that the expected Esscher transform will converge to the essential infimum of X from above, as α tends to minus infinity and to the essential supremum of X from below as α tends to plus infinity.

Remark 2. Theorems 1 and 2 are equivalent to each other in the sense that one could prove Theorem 2 first and then obtain a compact proof of Theorem 1 in the form of a corollary to Theorem 2. However, Theorem 1 is much more relevant to practical applications.

Remark 3. The expected Esscher transform is widely used in actuarial mathematics as a method of pricing of insurance contracts. There, it is known under name *Esscher premium*. Note that actuaries mainly restrict their attention to the Esscher transform of a non-negative random variable X with non-negative parameter α.

An alternative approach for proving results similar to those presented in this article can be found in the paper by Pratsiovytyi and Drozdenko (2016).

References

1. F. Esscher, *On the probability function in the collective theory of risk*, Skand. Aktuarietidskr. **15**, (1932), 175–195.
2. F. Esscher, *On approximate computations when the corresponding characteristic functions are known*, Skand. Aktuarietidskr. **46**, (1963), 78–86.
3. M. V. Pratsiovytyi and V. O. Drozdenko, *Limit behavior of the Esscher premium*, Random Oper. Stoch. Equ. **24** (2016), no. 2, 143–146.
4. H. Yang, *Esscher transform*, Encyclopedia of Actuarial Science (J. L. Teugels and B. Sundt, eds.), John Wiley & Sons, Chichester, 2004, 617–621.

Department of Higher Mathematics and Physics, Bila Tserkva National Agrarian University, 8/1 Soborna Sq., Bila Tserkva, 09117, Ukraine
Email: drozdenko0408@gmail.com

Voronoï's Impact on Modern Science, Book 6, Volume 1
Proceedings of the Sixth International Conference on Analytic Number Theory and Spatial Tessellations

CORRESPONDING N-DIMENSIONAL CONTINUED FRACTIONS FOR N-MULTIPLE POWER SERIES

Khrystyna Kuchminska (*Ukraine*)

Abstract. One of the possible algorithms for the development of a formal N-multiple power series into a functional N-dimensional regular C-fraction or associated continued fraction corresponding to this series is proposed. In this case, development algorithms into corresponding and two-dimensional corresponding fractions for the power and double power series are used.

Introduction

One of the approaches to represent analytic functions by continued fractions is the construction of a corresponding continued fraction. This formal continued fraction expansion is obtained by requiring that the Laurent expansion of the nth approximant agree term by term with a given Laurent series L up to the ν_n power of z, where z tends to infinity with n. Continued fractions defined in this way are said to correspond to the series L (or to the function $f(z)$ of which L is a Laurent expansion) [1].

In spite of the fact that Gauss, and later Stieltjes and Chebyshev got the development of power series in continued fractions, using the idea of correspondence, and also used such developments in applied problems, O. Perron introduced corresponding continued fractions for $\nu_n = n + 1$, $\nu_n = 2n + 1$, the general theory of correspondence was outlined much later by W. B. Jones, W. J. Thron in [1]. In particular, they consider algorithms for the development of functions into the regular C-fractions and associated continued fractions.

Two-dimensional corresponding continued fractions, as one of the constructive methods for the approximation of functions of two variables,

Received August 16, 2018; revised November 21, 2018.
2010 *Mathematics Subject Classification.* Primary 40A15; Secondary 30B70.
Key words and phrases. N-dimensional continued fraction, N-multiple power series, correspondence.

were introduced by Kh. Kuchminska [2], J. A. Murphy, M. R. O'Donohoe [3], and W. Siemaszko [2].

Main results

Consider a sequence of rational functions $\{R_k(z)\} = P_m(z)/Q_l(z)$, $z = (z_1, ..., z_N) \in \mathbb{C}^N$, $k = 1, 2, ...$, $degP_m(z) = m(N)$, $degQ_l(z) = l(N)$. Expand the function $R_k(z)$ into the formal N-multiple power series at $O = (0, 0, ..., 0)$. It's possible iff $Q_l(0) \neq 0$. Denote $T(R_k)$ the expansion of $R_k(z)$ into the N-multiple Taylor series at $O = (0, 0, ..., 0)$.

Definition 1. A sequence $\{R_k(z)\}$ will be said to correspond at the origin to a formal N-multiple Taylor series $P(z)$

$$P(z) = \sum_{|k|=0}^{\infty} c_k z^k, \quad z = (z_1, ..., z_N) \in \mathbb{C}^N, \quad k = (k_1, ..., k_N) \in \mathbb{Z}_+^N \qquad (1)$$

$$|k| = k_1 + ... + k_N, \quad , z^k = z_1^{k_1} ... z_N^{k_N},$$

if

$$\lim_{k\to\infty} \lambda(P - T(R_k(z))) = \infty, \quad \lambda(P) = \begin{cases} \infty, \ if \ P = 0, \\ s, \quad if \ P \neq 0, s = |k|. \end{cases}$$

If the sequence $\{R_k(z)\}$ corresponds to a formal N-multiple Taylor series $P(z)$, then the order of correspondence of $R_n(z)$ is defined to be

$$\nu_n = \lambda \left(P - T(R_k(z))\right).$$

Consider an N-dimensional continued fraction

$$\overset{\infty}{\underset{i=0}{D}} \frac{a_{\underbrace{i, i, \ldots, i}_{N}}(z)}{F_i(z)},$$

where $F_i(z)$ is the sum of p-dimensional continued fractions ($1 \leq p \leq N - 1$) with approximants

$$f_n(z) = \overset{n}{\underset{i=0}{D}} \frac{a_{\underbrace{i, i, \ldots, i}_{N}}(z)}{F_i^{(n-1)}(z)},$$

where $F_i^{(n-1)}(z)$ is the sum of the $(n-1)$th approximants of p-dimensional continued fractions ($1 \leq p \leq N - 1$).

Definition 2. The N-dimensional continued fraction corresponds to the formal N-multiple power series $P(z)$, if the sequence of its approximants corresponds to $P(z)$ at $O = (0, 0, ..., 0)$.

We denote

$$i\left[i_{p_1}, i_{p_2}, \ldots, i_{p_s}\right] =$$
$$= \left[i_1\left(i_{p_1}, i_{p_2}, \ldots, i_{p_s}\right), i_2\left(i_{p_1}, i_{p_2}, \ldots, i_{p_s}\right), \ldots, i_n\left(i_{p_1}, i_{p_2}, \ldots, i_{p_s}\right)\right],$$
$$p_s = 1, \ldots, N, s = 1, \ldots, N,$$
$$i_k\left(i_{p_1}, i_{p_2}, \ldots, i_{p_s}\right) = \begin{cases} i_{p_m},\ if\ k = p_m,\ m = 1, 2, \ldots, s \\ 0,\ \ if\ k \neq p_m \end{cases}$$

and

$$(i \pm t)\left[i_{p_1}, i_{p_2}, \ldots, i_{p_s}\right] =$$
$$= \left[i_1\left(i_{p_1}, i_{p_2}, \ldots, i_{p_s}\right) \pm t_1; \ldots; i_s\left(i_{p_1}, i_{p_2}, \ldots, i_{p_s}\right) \pm t_s; 0, \ldots, 0\right],$$

$$t = (t_1, t_2, \ldots, t_s), \quad 1 = (1, 1, \ldots, 1).$$

Then the formal N-multiple power series $P(z)$ can be represented in the form

$$c_0 + \sum_{p_1=1}^{N} \sum_{p_1 \geq 1} c_{i\left[i_{p_1}\right]} z_{p_1}^{i_{p_1}} + \sum_{p_1=1}^{N-1} \sum_{p_2=p_1+1}^{N} z_{p_1} z_{p_2} \sum_{i_{p_1}+i_{p_2} \geq 0} c_{(i+1)\left[i_{p_1}, i_{p_2}\right]} z_{p_1}^{i_{p_1}} z_{p_2}^{i_{p_2}} +$$

$$+ \sum_{p_1=1}^{N-2} \sum_{p_2=p_1+1}^{N-1} \sum_{p_3=p_1+2}^{N} z_{p_1} z_{p_2} z_{p_3} \sum_{i_{p_1}+i_{p_2}+i_{p_3} \geq 0} c_{(i+1)\left[i_{p_1}, i_{p_2}, i_{p_3}\right]} z_{p_1}^{i_{p_1}} z_{p_2}^{i_{p_2}} z_{p_3}^{i_{p_3}} + \ldots +$$

$$+ z_1 z_2 \ldots z_N \sum_{i_1+i_2+\ldots+i_N \geq 0} c_{(i+1)\left[i_{p_1}, i_{p_2}, \ldots, i_{p_N}\right]} z_1^{i_1} z_2^{i_2} \ldots z_N^{i_N}. \qquad (2)$$

The first N terms of series (2) are power series of one variable and can be expanded into corresponding regular C-fractions

$$1 + \cfrac{a_1 z}{1 + \cfrac{a_2 z}{1 + \ddots}} = 1 + \underset{i=1}{\overset{\infty}{D}} \frac{a_i z}{1} \ or \ \left(1 + \underset{i=1}{\overset{\infty}{D}} \frac{a_i z}{1}\right)^{-1}$$

with approximants

$$f_n(z) = 1 + \underset{i=1}{\overset{n}{D}} \frac{a_i z}{1} \ or \ f_n = \left(1 + \underset{i=1}{\overset{n-1}{D}} \frac{a_i z}{1}\right)^{-1}, \ f_0 = 1$$

or associated continued fractions

$$1 + \cfrac{k_1 z}{1 + l_1 z + \cfrac{-k_2 z^2}{1 + l_2 z + \cfrac{-k_3 z^2}{1 + l_3 z + \ddots}}} = 1 + \cfrac{k_1 z}{1 + l_1 z + \underset{i=2}{\overset{\infty}{D}} \cfrac{-k_i z^2}{1 + l_i z}}, k_i \neq 0$$

or

$$\left(1+\frac{k_1z}{1+l_1z+\underset{i=2}{\overset{\infty}{D}}\frac{-k_iz^2}{1+l_iz}}\right)^{-1},k_i\neq 0$$

with approximants

$$f_n(z)=1+\frac{k_1z}{1+l_1z+\underset{i=2}{\overset{n}{D}}\frac{-k_iz^2}{1+l_iz}},\ k_i\neq 0,\ f_0=1,\ f_1=1+\frac{k_1z}{1+l_1z}$$

or with approximants

$$f_n=\left(1+\frac{k_1z}{1+l_1z+\underset{i=2}{\overset{n}{D}}\frac{-k_iz^2}{1+l_iz}}\right)^{-1},\quad k_i\neq 0,$$
$$f_1=1,\quad f_2=\left(1+\frac{k_1z}{1+l_1z}\right)^{-1}.$$

There are well-known formulas to obtain coefficients for these corresponding fractions by coefficients of the power series [1].

The next $N-1$ terms can be developed into two-dimensional continued regular C-fractions [2]:

$$\Phi_0(z_1,z_2)+\underset{i=1}{\overset{\infty}{D}}\frac{a_{i,i}z_1z_2}{\Phi_i(z_1,z_2)}\ \ or\ \ \left(\Phi_0(z_1,z_2)+\underset{i=1}{\overset{\infty}{D}}\frac{a_{i,i}z_1z_2}{\Phi_i(z_1,z_2)}\right)^{-1},$$

where

$$\Phi_0(z_1,z_2)=a_{0,0}+\underset{k=1}{\overset{\infty}{D}}\frac{a_{k,0}z_1}{1}+\underset{k=1}{\overset{\infty}{D}}\frac{a_{0,k}z_2}{1},$$

and

$$\Phi_i(z_1,z_2)=1+\underset{k=1}{\overset{\infty}{D}}\frac{a_{i+k,i}z_1}{1}+\underset{k=1}{\overset{\infty}{D}}\frac{a_{i,i+k}z_2}{1},\ i=1,2,\dots,$$

$a_{i,j}\neq 0$. $i=0,1,\dots,\ j=0,1,\dots$ with approximants

$$f_n(z_1,z_2)=\frac{A_n(z_1,z_2)}{B_n(z_1,z_2)}=\Phi_0^{(n)}(z_1,z_2)+\underset{i=1}{\overset{n}{D}}\frac{a_{i,i}z_1z_2}{\Phi_i^{(n-i)}(z_1,z_2)},\quad f_0=1$$

or

$$f_n(z_1,z_2)=\left(\Phi_0^{(n-1)}(z_1,z_2)+\underset{i=1}{\overset{(n-1)}{D}}\frac{a_{i,i}z_1z_2}{\Phi_i^{(n-1-i)}(z_1,z_2)}\right)^{-1},\ f_0=1,$$

where

$$\Phi_i^{(m)}(z_1,z_2)=1+\underset{k=1}{\overset{m}{D}}\frac{a_{i+k,i}z_1}{1}+\underset{k=1}{\overset{m}{D}}\frac{a_{i,i+k}z_2}{1},\ i=1,2,\dots,\ \Phi_i^{(0)}=1.$$

Similar developments of $N-1$ terms of the series (2) can be written in the associated two-dimensional continued fraction form [2]:

$$\Psi_0(z_1,z_2)+\underset{i=1}{\overset{\infty}{D}}\frac{a_{i,i}z_1z_2}{\Psi_i(z_1,z_2)}\ \ or\ \ \left(\Psi_0(z_1,z_2)+\underset{i=1}{\overset{\infty}{D}}\frac{a_{i,i}z_1z_2}{\Psi_i(z_1,z_2)}\right)^{-1},$$

where

$$\Psi_0(z_1,z_2)=$$

$$=a_{0,0}+\frac{k_{1,0}^{(0)}z_1}{1+l_{1,0}^{(0)}z_1-\underset{j=2}{\overset{\infty}{D}}\frac{k_{j,0}z_1^2}{1+l_{j,0}^{(0)}z_1}}+\frac{k_{0,1}^{(0)}z_2}{1+l_{0,1}^{(0)}z_2-\underset{j=2}{\overset{\infty}{D}}\frac{k_{0,j}z_2^2}{1+l_{0,j}^{(0)}z_2}}$$

and

$$\Psi_i(z_1,z_2)=1+\frac{(-1)^ik_{1,0}^{(i)}z_1}{1+(-1)^il_{1,0}^{(i)}z_1-\underset{j=2}{\overset{\infty}{D}}\frac{k_{j,0}z_1^2}{1+(-1)^il_{j,0}^{(i)}z_1}}+$$

$$+\frac{(-1)^ik_{0,1}^{(i)}z_2}{1+(-1)^il_{0,1}^{(i)}z_2-\underset{j=2}{\overset{\infty}{D}}\frac{k_{0,j}z_2^2}{1+(-1)^il_{0,j}^{(i)}z_2}}.$$

Coefficients are obtained by coefficients of the power series. Approximants and coefficients for these fractions can be found in [2].

The rest $\sum\limits_{i=3}^{N}\frac{N!}{(N-i)!i!}$ terms can be transformed in the following way:

$$\sum_{p_1=1}^{N-s+1}\sum_{p_2=p_1+1}^{N-s}\dots\sum_{p_s=p_1+s-1}^{N}z_{p_1}z_{p_2}\dots z_{p_s}c_{(i+1)[\underbrace{0,\dots,0}_{s}]}\times$$

$$\times\sum_{i_{p_1}+i_{p_2}+\dots,i_{p_s}\geq 0}\frac{c_{(i+1)\left[i_{p_1},i_{p_2},\dots,i_{p_s}\right]}}{c_{(i+1)[\underbrace{0,\dots,0}_{s}]}}z_{p_1}^{i_{p_1}}z_{p_2}^{i_{p_2}}\dots z_{p_s}^{i_{p_s}}=$$

$$\sum_{p_1=1}^{N-s+1}\sum_{p_2=p_1+1}^{N-s}\cdots\sum_{p_s=p_1+s-1}^{N}\frac{c_{(i+1)[\underbrace{0.\ldots,0}_{s}]}z_{p_1}z_{p_2}\cdots z_{p_s}}{1}$$

$$\frac{1}{\sum\limits_{i_{p_1}+i_{p_2}+\ldots,i_{p_s}\geq 0}\frac{c_{(i+1)[i_{p_1},i_{p_2},\ldots,i_{p_s}]}}{c_{(i+1)[\underbrace{0.\ldots,0}_{s}]}}z_{p_1}^{i_{p_1}}z_{p_2}^{i_{p_2}}\ldots z_{p_s}^{i_{p_s}}}$$

$$=\sum_{p_1=1}^{N-s+1}\sum_{p_2=p_1+1}^{N-s}\cdots\sum_{p_s=p_1+s-1}^{N}c_{(i+1)[\underbrace{0.\ldots,0}_{s}\ldots,0]}z_{p_1}z_{p_2}\cdots z_{p_s}\times$$

$$\left(\sum_{i_{p_1}+i_{p_2}+\ldots,i_{p_s}\geq 0}(-1)^{i_{p_1}+i_{p_2}+\ldots+i_{p_s}}\gamma^{(1)}_{i\left[i_{p_1},i_{p_2},\ldots,i_{p_s}\right]}z_1^{i_1}z_2^{i_2}\ldots z_N^{i_N}\right)^{-1}.$$

Coefficients $\gamma^{(1)}_{i\left[i_{p_1},i_{p_2},\ldots,i_{p_s}\right]}$ are uniquely determined from the following recurrence relations:

$$\gamma^{(1)}_{i\left[i_{p_1},i_{p_2},\ldots,i_{p_s}\right]}=$$

$$\sum_{t_1+t_2+\ldots+t_s=1}^{i_{p_1}+i_{p_2}+\ldots+i_{p_s}}(-1)^{t_1+t_2+\ldots+t_s-1}\gamma^{(1)}_{(i-t)\left[i_{p_1},i_{p_2},\ldots,i_{p_s}\right]}\frac{c_{(i+1)[t_1,t_2,\ldots,t_s]}}{c_{(i+1)[0,0,\ldots,0]}},$$

where

$$\gamma^{(1)}_0=1,\ \gamma^{(1)}_{(i-t)\left[i_{p_1},i_{p_2},\ldots,i_{p_s}\right]}=0, i_m\left[i_{p_1},i_{p_2},\ldots,i_{p_s}\right]-t_m<0,$$

$$m=1,\ldots,s;\ i_k(t_1,t_2,\ldots,t_s)=\begin{cases}t_k,\ if\ s=k,\\ 0,\ \ if\ s\neq k.\end{cases}$$

Then the power series

$$\sum_{i_{p_1}+i_{p_2}+\ldots+i_{p_s}\geq=0}(-1)^{i_{p_1}+i_{p_2}+\ldots+i_{p_s}}\gamma^{(1)}_{i\left[i_{p_1},i_{p_2},\ldots,i_{p_s}\right]}z_{p_1}^{i_{p_1}}\ldots z_{p_s}^{i_{p_s}}$$

can be represented in the form

$$\gamma^{(1)}_0+\sum_{p_1=1}^{N}\sum_{i_{p_1}\geq 1}(-1)^{i_{p_1}}\gamma^{(1)}_{i\left[i_{p_1}\right]}z_{p_1}^{i_{p_1}}+$$

$$\sum_{p_1=1}^{N-1}\sum_{p_2=p_1+1}^{N}z_{p_1}z_{p_2}\sum_{i_{p_1}+i_{p_2}\geq 0}(-1)^{i_{p_1}+i_{p_2}}\gamma^{(1)}_{(i+1)\left[i_{p_1},i_{p_2}\right]}z_{p_1}^{i_{p_1}}z_{p_2}^{i_{p_2}}+\ldots+$$

$$z_1z_2\ldots z_N\sum_{i_1+i_2+\ldots+i_{\mathrm{N}}\geq 0}(-1)^{i_{p_1}+i_{p_2}+\ldots+i_{p_N}}\gamma^{(1)}_{(i+1)\left[i_{p_1},i_{p_2},\ldots,i_{p_{\mathrm{N}}}\right]}z_1^{i_1}z_2^{i_2}\ldots z_N^{i_N}. \tag{3}$$

The first N terms of the series (3) are the power series of one variable and can be expanded into corresponding regular C-fractions or associated continued fractions. In the other terms, an algorithm similar to the previous one can be applied.

Now, the necessary and sufficient conditions for the development of a formal N-multiple power series into a functional N-dimensional regular C-fraction or an associated fraction corresponding to this series can be formulated.

Theorem 1. *N-multiple power series* (1) *develops into the corresponding N-dimensional regular C-fraction* (*or the associated N-dimensional continued fraction*) *if and only if all determinants* $\phi^{(k)}_{mi[i_{p_s}]}$, $\psi^{(k)}_{mi[i_{p_s}]}$, $k = 0, 1\ldots,$ $i_{p_s} = 1,$ $p_s = 1, \ldots, N,$ $s = 1, \ldots, N,$ $m = 0, 1, \ldots$ (*or* $\phi^{(k)}_{mi[i_{p_s}]}$, $k = 0, 1\ldots,$ $i_{p_s} = 1,$ $p_s = 1, \ldots, N,$ $s = 1, \ldots, N,$ $m = 0, 1, \ldots$) *are different from zero:*

$$\phi^{(k)}_{mi[i_{p_s}]} = \begin{vmatrix} \gamma^{(k)}_{i[i_{p_s}]} & \gamma^{(k)}_{(i+1)[i_{p_s}]} & \cdots & \gamma^{(k)}_{(i+m-1)[i_{p_s}]} \\ \gamma^{(k)}_{(i+1)[i_{p_s}]} & \gamma^{(k)}_{(i+2)[i_{p_s}]} & \cdots & \gamma^{(k)}_{(i+m)[i_{p_s}]} \\ \cdots & \cdots & \cdots & \cdots \\ \gamma^{(k)}_{(i+m-1)[i_{p_s}]} & \gamma^{(k)}_{(i+m)[i_{p_s}]} & \cdots & \gamma^{(k)}_{(i+2m-2)[i_{p_s}]} \end{vmatrix}, m = 1, 2, \ldots, \tag{4}$$

$$\psi^{(k)}_{mi[i_{p_s}]} = \begin{vmatrix} \gamma^{(k)}_{(i+1)[i_{p_s}]} & \gamma^{(k)}_{(i+2)[i_{p_s}]} & \cdots & \gamma^{(k)}_{(i+m-1)[i_{p_s}]} \\ \gamma^{(k)}_{(i+2)[i_{p_s}]} & \gamma^{(k)}_{(i+3)[i_{p_s}]} & \cdots & \gamma^{(k)}_{(i+m)[i_{p_s}]} \\ \cdots & \cdots & \cdots & \cdots \\ \gamma^{(k)}_{(i+m-1)[i_{p_s}]} & \gamma^{(k)}_{(i+m)[i_{p_s}]} & \cdots & \gamma^{(k)}_{(i+2m-3)[i_{p_s}]} \end{vmatrix}, m = 1, 2, \ldots, \tag{5}$$

where

$$\phi_{0i[i_{p_s}} = \psi_{1i[i_{p_s}} = 1,\ \gamma^{(0)}_{i[i_{p_s}} = c_{i[i_{p_s}},\ \gamma^{(k)}_0 = 1,$$

$$\gamma^{(k)}_{i\left[i_{p_1}, i_{p_2}, \ldots, i_{p_s}\right]} =$$

$$\sum_{t_1+t_2+\ldots+t_s=1}^{i_{p_1}+i_{p_2}+\ldots+i_{p_s}} (-1)^{t_1+t_2+\ldots+t_s-1} \gamma^{(k)}_{(i-t)\left[i_{p_1}, i_{p_2}, \ldots, i_{p_s}\right]} \frac{\gamma_{(i+1)[t_1, t_2, \ldots, t_s]}}{\gamma_{(i+1)[0,0,\ldots,0]}},$$

$\gamma^{(k)}_{(i-t)\left[i_{p_1}, i_{p_2}, \ldots, i_{p_s}\right]} = 0$ *if* $i_m\left[i_{p_1}, i_{p_2}, \ldots, i_{p_s}\right] - t_m < 0,\ m = 1, \ldots, s;$

$$i_k(t_1, t_2, \ldots, t_s) = \begin{cases} t_k, & if\ s = k, \\ 0, & if\ s \neq k \end{cases}$$

Proof. The method of complete mathematical induction, the difference from the zero determinants (4), (5), which allows to find the coefficients of the fractions by coefficients of the series (1), as well as the outgoing results for continued and two-dimensional continued fractions, are used substantially to prove the theorem. □

References

1. W. B. Jones and W. J. Thron, *Continued fractions: analytic theory and applications*, Addison-Wesley Publ., Mass., 1980.
2. Kh. Kuchminska, *Two-dimensional continued fractions*, Pidstryhach Institute for Applied Problems in Mechanics and Mathematics of the NAS of Ukraine, Lviv, 2010 (in Ukrainian).
3. J. A. Murphy and M. R. O'Donohoe, *A two-variable generalization of the Stieltjes-type continued fraction*, J. Comput. Appl. Math. **4** (1978), no. 3, 181–190.

Department of Differential Equations and Function Theory, Pidstryhach Institute for Applied Problems in Mechanics and Mathematics of the NAS of Ukraine, 3-B Naukova St., Lviv, 79060, Ukraine
Email: khkuchminska@gmail.com

Author Index

Boytsun, Liliya, 146
Brukh, Roman, xii

Drozdenko, Vitaliy, 159

Gefter, Sergiy, 87
Goy, Taras, 41

Kuchminska, Khrystyna, 169

Laohakosol, Vichian, 1, 48, 65, 138
Lelechenko, Andrew V., 76

Martseniuk, Vsevolod, 87
Mavecha, Sukrawan, 138

Oliynyk, Bogdana, 24

Pabhapote, Nittiya, 48
Piven', Aleksey, 87
Pratsiovytyi, Mykola, viii

Ruengsinsub, Pattira, 1
Rybnikova, Tamara, 146

Savastru, Olga, 96
Spivakovskyi, Oleksandr, x
Srichan, Teerapat, 33, 107
Sripayap, Angkana, 65
Steuding, Jörn, vii, 33

Tóth, László, 40

Vaiginytė, Adelė, 120
Varbanets, Pavel, 124
Varbanets, Sergey, 124

Yuttanan, Boonrod, 138

Наукове видання

ВПЛИВ ВОРОНОГО НА СУЧАСНУ НАУКУ

Праці
Шостої міжнародної конференції
з аналітичної теорії чисел
і просторових мозаїк

У двох томах
том 1

Редактори Йорн Штойдинґ,
Микола Працьовитий
Упорядник Галина Сита
Комп'ютерна верстка Олександр Барановський

Підписано до друку 17.12.2018. Формат 70 × 108/16.
Гарнітура Computer Modern.
Ум. др. арк. 16,98. Обл.-вид. арк. 10,05.
Тираж 150 прим. Зам. № 5507.
Віддруковано з оригінал-макету.

Видавництво Національного педагогічного університету
імені М. П. Драгоманова. 01601, м. Київ-30, вул. Пирогова, 9
Свідоцтво про реєстрацію ДК № 1101 від 29.10.2002. (044) 234-75-87

Віддруковано ВД «Академперіодика» НАН України
01004, м. Київ, вул. Терещенківська, 4
Свідоцтво про внесення до Державного реєстру
суб'єкта видавничої справи серії ДК № 544 від 27.07.2001