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Classical Gasses with Singular Densities

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Abstract: We investigate classical continuous systems characterized by singular velocity distributions, where the corresponding Radon measures are defined over the entire space with infinite mass. These singular distributions are used to model particle velocities in systems where traditional velocity distributions do not apply. As a result, the particle positions in such systems no longer conform to conventional configurations in physical space. This necessitates the development of novel analytical tools to understand the underlying models. To address this, we introduce a new conceptual framework that redefines particle configurations in phase space, where each particle is represented by its spatial position and a velocity vector. The key idea is the construction of the *Plato space*, which is designed to represent idealized particle configurations where the total velocity remains bounded within any compact subset of phase space. This space serves as a crucial bridge to the space of vector-valued discrete Radon measures, where each measure captures the velocity distribution over the entire system. Given the inherent complexity of analyzing infinite-dimensional spaces, we tackle the problem by reformulating it onto a finite-dimensional configuration space. This is achieved by decomposing the infinite space into smaller, more manageable components. A central tool in this reformulation is the *K-transform*, which is pivotal in enabling harmonic analysis of the space. The K-transform allows us to represent the system in terms of components that are more amenable to analysis, thus simplifying the study of the system's dynamics. Furthermore, we extend previous results in the study of correlation functions by developing correlation measures tailored for these vector-valued Radon measures. These generalized functions provide deeper insights into the correlations between particle positions and velocities, expanding the range of analysis to systems with singular velocity distributions. Through this approach, we develop a robust mathematical framework that sheds light on the structure and dynamics of complex particle systems, especially those characterized by singular velocity distributions. Our results offer a new perspective on systems with non-traditional velocity distributions, advancing the theory and methodology of particle systems in both classical and modern contexts.

Keywords: classical gasses; Poisson point process; correlation functions; discrete measures

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1. Introduction

Let $\Gamma(\mathbb{R}^d_0 \times \mathbb{R}^d)$ denote the space of configurations for an interacting particle system, where each particle is described by a pair (v_x, x), with $v_x \in \mathbb{R}^d_0 = \mathbb{R}^d \setminus \{0\}$ representing the velocity and $x \in \mathbb{R}^d$ the position. The phase space is then $\mathbb{R}^d_0 \times \mathbb{R}^d$, and locally finite subsets of this space give the configurations. Specifically, a configuration $\gamma \in \Gamma(\mathbb{R}^d_0 \times \mathbb{R}^d)$ is a set of pairs $\gamma = \{(v_x, x)\} \subset \mathbb{R}^d_0 \times \mathbb{R}^d$, where each particle is characterized by its velocity v_x and position x.

We focus on a subset of configurations, denoted by $\Gamma_p(\mathbb{R}_0^d \times \mathbb{R}^d)$, called *pinpointed configurations*. A configuration is pinpointed if no two particles are positioned in the same location in $\mathbb{R}^d$ unless they have the same velocity, i.e., each position x in the configuration corresponds to at most one velocity v_x.

Further, we define the *Plato space*, denoted by $\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)$, as the subset of $\Gamma_p(\mathbb{R}_0^d \times \mathbb{R}^d)$ consisting of configurations where the total velocity within any compact domain of $\mathbb{R}^d$ is bounded. This constraint ensures that the system's behavior remains physically plausible by avoiding the occurrence of unbounded velocities within finite spatial domains. We define the cone of vector-valued discrete Radon measures $\mathbb{K}(\mathbb{R}^d)$ as a subset of the space of Radon measures $\mathcal{M}(\mathbb{R}^d)$:

$$\mathbb{K}(\mathbb{R}^d) = \left\{ \eta = \sum_{x \in \tau(\eta)} v_x \delta_x \in \mathcal{M}(\mathbb{R}^d) \middle| x \in \mathbb{R}^d,\ v_x \in \mathbb{R}_0^d \right\},$$

where δ_x denotes the Dirac measure with a unit mass at $x \in \mathbb{R}^d$ and the support of η is defined as

$$\tau(\eta) = \{x \in \mathbb{R}^d \mid \eta(\{x\}) =: v_x(\eta)\}.$$

This paper is dedicated to the in-depth exploration and analysis of the cone $\mathbb{K}(\mathbb{R}^d)$, which serves as a significant framework for modeling particle systems in real-world scenarios. However, defining mathematical structures in $\mathbb{K}(\mathbb{R}^d)$ presents substantial challenges. To address these, we employ the Plato space $\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)$, where the elements of this space, termed ideas, are mapped through the reflection mapping $\mathcal{R}$ to corresponding elements in $\mathbb{K}(\mathbb{R}^d)$, which represent observed objects (i.e., images of ideas). The properties of the object space in $\mathbb{K}(\mathbb{R}^d)$ are intrinsically tied to the structure of the idea space in the Plato space. This conceptual framework is inspired by Plato's theory of forms, which asserts that the observable world is a projection of a higher realm of ideal forms for more details (see [1] for further details). The relationship between the Plato space $\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)$ and the cone of vector-valued discrete Radon measures $\mathbb{K}(\mathbb{R}^d)$, mediated by reflection mapping, forms the basis for the analysis of $\mathbb{K}(\mathbb{R}^d)$.

The nonlinearity and complexity of the infinite-dimensional spaces involved present significant challenges in directly analyzing the dynamics modeled on $\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)$. To overcome this, we propose reinterpreting the equations within the finite configuration space $\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)$, a subset of $\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)$, where the finite configuration space $\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)$ is decomposed using n-point configurations. Details of this construction are provided in Definition 2. This reformulation simplifies the analysis, and the K-transform plays a central role in enabling this transition. The defining feature of this transform is its inherently combinatorial nature, which serves as the foundation for the development of combinatorial harmonic analysis of configuration spaces. Notably, the operator nature of the K-transform was first identified by Lenard [2–4] in the context of specific problems in statistical mechanics.

Harmonic analysis of configuration spaces over Riemannian manifolds has been extensively developed in [5], primarily focusing on particle systems in the continuum, such as those defined in $\mathbb{R}^d$. These systems consider particles distributed continuously across the space, with no discrete constraints, enabling the study of dynamic configurations in a continuous setting. The approach allows for the examination of the behavior of particle distributions, their interactions, and the evolution of such systems under various conditions. In [6,7], the analysis was extended to encompass interacting particle systems within the continuum $\mathbb{R}^d$, introducing the concept of "marks". In these frameworks, each particle $x \in \mathbb{R}^d$ is assigned a positive scalar mark, typically representing additional properties such as charge, mass, or other intrinsic characteristics that affect the system's dynamics. These marks provide additional structure to the configuration space, facilitating the study of systems where the particles are not only distributed across the space but also distinguished by their specific attributes. Our work extends this framework further by considering particle systems in $\mathbb{R}^d$ where, instead of scalar marks, each particle is associated with a

velocity vector $v_x \in \mathbb{R}_0^d$, with $\mathbb{R}_0^d$ representing the d-dimensional Euclidean space with the exclusion of the zero vector (i.e., $v_x \neq 0$). This transition from scalar- to vector-valued marks introduces a significantly more complex structure, as the system must now account for both the spatial configuration of the particles and their associated velocities. The velocities introduce a dynamical aspect to the configuration space, linking the positions and motions of the particles more intricately. To rigorously model such systems, we define the cone of vector-valued discrete Radon measures, denoted $\mathbb{K}(\mathbb{R}^d)$. This cone represents a set of measures describing particles' distribution in $\mathbb{R}^d$ alongside their velocities. Specifically, an element of $\mathbb{K}(\mathbb{R}^d)$ is a measure that assigns a mass to each configuration of particle positions and associated velocities, ensuring that both the spatial and velocity components are treated consistently within the same framework. The measures in $\mathbb{K}(\mathbb{R}^d)$ are discrete in nature, as they correspond to particle systems with a finite or countably infinite number of particles, each associated with a specific position and velocity. The structure of $\mathbb{K}(\mathbb{R}^d)$ is particularly powerful, as it encapsulates the spatial distribution of particles and the distribution of their velocities. This allows for a unified approach to studying the dynamics of such systems, including the analysis of transport, interactions, and statistical properties. In particular, this framework enables the exploration of the relationships between the spatial and velocity components of a system, leading to insights into the system's evolution over time. The cone structure also facilitates the formulation of variational principles and optimization problems that arise when studying the dynamics of interacting particle systems in $\mathbb{R}^d$. These principles, when applied to $\mathbb{K}(\mathbb{R}^d)$, provide a robust foundation for understanding the equilibrium states and transport phenomena in systems of particles with both spatial and velocity attributes. Furthermore, the mathematical framework of $\mathbb{K}(\mathbb{R}^d)$ opens avenues for studying statistical mechanics, thermodynamics, and fluid dynamics in the context of interacting particle systems. By treating the system as a measure in the product space of positions and velocities, we can study macroscopic phenomena such as the evolution of particle densities, the transport of momentum, and the impact of interactions on the overall system's dynamics. In this way, our approach builds upon the foundations laid in previous works but introduces a richer and more complex analysis that accounts for both the spatial and velocity components of particle systems in the continuum.

In this paper, we perform harmonic analysis in the space $\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)$ and extend this analysis to the cone of vector-valued discrete Radon measures $\mathbb{K}(\mathbb{R}^d)$ using the reflection mapping $\mathcal{R}$. The mapping $\mathcal{R}$ is defined as follows:

$$\mathcal{R} : \Pi(\mathbb{R}_0^d \times \mathbb{R}^d) \to \mathbb{K}(\mathbb{R}^d),$$

$$\gamma = \sum_{(v_x,x)\in\gamma} \delta_{(v_x,x)} \to \sum_{x\in\tau(\gamma)} v_x \delta_x.$$

Additionally, we investigate the correlation measures and correlation functions associated with probability measures in the cone.

While the papers [5–7] provided foundational results for scalar-valued discrete Radon measures, the present work extends these results to the vector-valued setting. Specifically, we introduce the cone of vector-valued discrete Radon measures $K(\mathbb{R}^d)$, where each measure takes the following form:

$$\eta = \sum_{x\in\tau(\eta)} v_x \delta_x, \quad \text{with } v_x \in \mathbb{R}_0^d.$$

This extension significantly increases the complexity, as it incorporates directional dynamics and interactions that are not present in the scalar case. Using vector-valued measures allows us to capture both the position and velocity of particles, providing a richer framework for modeling systems with singular velocity distributions. Moreover, a novel

conceptual tool introduced in this paper is the Plato space $\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)$, which consists of configurations where the total velocity within any compact subset is bounded. Formally,

$$\Pi(\mathbb{R}_0^d \times \mathbb{R}^d) = \left\{ \gamma \in \Gamma_p(\mathbb{R}_0^d \times \mathbb{R}^d) \mid V_\Lambda(\gamma) = \sum_{x \in \tau(\gamma) \cap \Lambda} |v_x| < \infty, \ \forall \Lambda \in \mathcal{B}_c(\mathbb{R}^d) \right\}.$$

The reflection mapping $\mathcal{R} : \Pi(\mathbb{R}_0^d \times \mathbb{R}^d) \to K(\mathbb{R}^d)$ acts as a bridge between the Plato space and the cone of vector-valued measures:

$$\mathcal{R}(\gamma) = \sum_{x \in \tau(\gamma)} v_x \delta_x.$$

This mapping enables us to transfer the structure and properties of the Plato space to the space of vector-valued measures, facilitating the analysis of complex systems. It is also worth mentioning that, to handle the challenges posed by infinite-dimensional spaces, we decompose the Plato space into finite configurations $\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)$ and introduce the K-transform. This transform maps functions on finite configurations to functions in the Plato space:

$$(KG)(\gamma) = \sum_{\xi \subset \gamma} G(\xi),$$

where G is a measurable function in finite configurations. The K-transform is crucial for enabling harmonic analysis in these spaces, facilitating the study of dynamics and correlations. In contrast to the scalar case, the vector-valued setting introduces additional challenges in defining and analyzing the K-transform due to the directional nature of the velocities. Moreover, we also extended the concept of correlation measures and n-point correlation functions to the vector-valued setting. The correlation measure ρ_μ associated with a probability measure μ in $\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)$ is defined as follows:

$$\rho_\mu(A) = \int_{\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)} K(A, \gamma) \mu(d\gamma),$$

where A is a measurable subset of finite configurations. The corresponding n-point correlation function $k_\mu^{(n)}$ characterizes the joint distribution of positions and velocities for n-particle systems. These generalizations provide a deeper understanding of interactions in systems with singular velocity distributions, which have not been fully addressed in prior studies. According to the introduction of the Plato space and reflection mapping, we provide novel insights into the structure of particle systems with singular velocities. By reformulating the problem in terms of vector-valued measures and finite decompositions, we are able to achieve the following: (1) analyze the interplay between spatial and velocity components of particle configurations; (2) develop variational principles and harmonic analysis tools specific to the vector-valued context; (3) extend integration-by-parts formulas and Dirichlet forms to systems with vector-valued measures, expanding the scope of stochastic analysis. These advancements go beyond scalar results, providing a comprehensive framework for studying more complex, real-world systems. In particular, it addresses the following key points:

- Many real-world phenomena involve particles with velocity distributions that exhibit singularities, such as in turbulence, plasma physics, and astrophysical systems. Classical models that assume finite mass or bounded velocities fail to capture these behaviors. Our framework, which utilizes vector-valued Radon measures with infinite mass and singular distributions, provides the necessary tools to analyze these systems rigorously. By introducing the Plato space $\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)$ and linking it to a cone $\mathbb{K}(\mathbb{R}^d)$ via reflection mapping $\mathcal{R}$, we enable the study of systems where velocities can become arbitrarily large while maintaining control over the total velocity in any compact domain.

- The complexity of infinite-dimensional configuration spaces has long posed a challenge in fields such as statistical mechanics, stochastic processes, and mean-field games. Our approach addresses this by decomposing infinite-dimensional space into finite, manageable components. The reformulation of problems in finite configurations, facilitated by the K-transform, allows for the practical computation and analysis of previously intractable systems. This decomposition is particularly useful for numerical simulations and approximate solutions in high-dimensional spaces, which are common in applied sciences.
- By extending harmonic analysis to vector-valued measures, our study provides new tools for analyzing particle systems with spatial and velocity components. The K-transform plays a central role in this advancement, enabling the decomposition of complex configurations into simpler substructures. This extension is useful for studying correlations, interactions, and dynamic behaviors in systems where particles are not merely points in space but carry additional vector-valued information, such as velocity, momentum, or force vectors.
- Our framework has potential applications in several modern scientific and engineering domains such as the following. (1) Kinetic theory: Indeed, the ability to handle singular velocity distributions is critical in the kinetic theory of gasses, especially when dealing with systems far from equilibrium, such as those involving shock waves or rarefied gasses. (2) Turbulence and fluid dynamics: This is because turbulent flows often exhibit velocity fields with singular behavior. Our approach provides a rigorous mathematical foundation for modeling and analyzing such flows, contributing to a better understanding and prediction of turbulence phenomena. (3) Astrophysics and cosmology: Indeed, in astrophysical systems, particles (e.g., stars, galaxies) can have highly irregular and unbounded velocities. Our framework enables the study of such systems using measures that can accommodate infinite mass distributions and singularities. (4) Mean-field games (MFGs): the study of interacting particle systems with vector-valued states has direct implications for mean-field games, where the behavior of large populations of agents is modeled. Accordingly, our framework provides new tools for analyzing equilibrium states and dynamics in MFGs with velocity-dependent interactions. (5) The unified treatment of spatial and velocity dynamics: This is implied by the fact that our work offers a unified mathematical approach to studying the spatial distribution of particles and their associated velocities. By framing the problem in terms of vector-valued discrete Radon measures, we facilitate the simultaneous analysis of particle positions and velocities. This unified approach is essential for applications where the interplay between position and velocity determines a system's evolution, such as in transport phenomena, optimization problems, and non-equilibrium statistical mechanics. (6) Insights into correlation structures: This follows from the fact that the extension of correlation measures and functions to vector-valued cases provides deeper insights into interactions and dependencies within complex particle systems. These insights are valuable for understanding clustering, dispersion, and collective behavior in systems ranging from physical particles to agents in multi-agent systems.

2. Results

2.1. Vector-Valued Random Discrete Measures

2.1.1. From Configuration Spaces to Discrete Measures

Consider the Euclidean space $\mathbb{R}^d$ and the locally compact pointed space $\mathbb{R}_0^d = \mathbb{R}^d \setminus \{0\}$. Let $\Gamma(\mathbb{R}_0^d \times \mathbb{R}^d)$ denote the configuration space over $\mathbb{R}_0^d \times \mathbb{R}^d$, i.e., the set of all locally finite subsets $\gamma = \{(v, x)\} \subset \mathbb{R}_0^d \times \mathbb{R}^d$. A subset is considered locally finite if it contains only finitely many points within any compact set. We equip this space with the vague topology, as described in [8], and the corresponding σ-algebra of subsets is denoted by $\mathcal{B}(\Gamma(\mathbb{R}_0^d \times \mathbb{R}^d))$.

Within all configurations, we focus on a specific subspace $\Gamma_p(\mathbb{R}_0^d \times \mathbb{R}^d)$, referred to as the space of pinpointed configurations. By definition, $\Gamma_p(\mathbb{R}_0^d \times \mathbb{R}^d)$ consists of configurations $\gamma \in \Gamma(\mathbb{R}_0^d \times \mathbb{R}^d)$ satisfying the condition that if $(v_1, x_1), (v_2, x_2) \in \gamma$ and $(v_1, x_1) = (v_2, x_2)$ with $x_1 = x_2$ then $v_1 = v_2$. This means that we cannot have two points with the same location in $\mathbb{R}^d$ but different values in $\mathbb{R}_0^d$. As easily seen, $\Gamma_p(\mathbb{R}_0^d \times \mathbb{R}^d) \in \mathcal{B}(\Gamma(\mathbb{R}_0^d \times \mathbb{R}^d))$. For any $\gamma \in \Gamma_p(\mathbb{R}_0^d \times \mathbb{R}^d)$, we can write the configuration as

$$\gamma = \{(v_x, x)\},\ \tau(\gamma) = \{x \mid (v_x, x) \in \gamma\} \subset \mathbb{R}^d,$$

where $\tau(\gamma)$ denotes the projection of the configuration onto its spatial positions in $\mathbb{R}^d$. We can also examine smaller subsets of configurations where $\tau(\gamma)$ remains a configuration. These subsets correspond to *marked point processes*, which offer a more tractable framework for study and analysis.

For any configuration $\gamma \in \Gamma_p(\mathbb{R}_0^d \times \mathbb{R}^d)$ and any compact set $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, we define the local velocity functional as follows:

$$V_\Lambda(\gamma) = \sum_{x \in \tau(\gamma) \cap \Lambda} |v_x| \leq \infty.$$

This functional measures the sum of the magnitudes of velocities associated with points in γ whose spatial positions lie within the compact set Λ. Using this functional, we define the Plato space $\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)$ as a subset of the pinpointed configuration space $\Gamma_p(\mathbb{R}_0^d \times \mathbb{R}^d)$. Specifically,

$$\Pi(\mathbb{R}_0^d \times \mathbb{R}^d) = \{\gamma \in \Gamma_p(\mathbb{R}_0^d \times \mathbb{R}^d)\ \forall \Lambda \in \mathcal{B}_c(\mathbb{R}^d) \mid V_\Lambda(\gamma) < \infty\}.$$

The corresponding trace σ-algebra is defined as $\mathcal{B}(\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)) := \{A \cap \Pi(R_0^d \times \mathbb{R}^d)) \mid A \in \mathcal{B}(\Gamma(R_0^d \times \mathbb{R}^d))\}$. Let $\mathcal{M}(\mathbb{R}^d, \mathbb{R}^d)$ denote the set of all vector-valued Radon measures defined in the Borel subsets $\mathcal{B}(\mathbb{R}^d)$, such that

$$\mathcal{M}(\mathbb{R}^d, \mathbb{R}^d) \ni \mu : \mathcal{B}(\mathbb{R}^d) \to \mathbb{R}^d,\ \forall \Lambda \in \mathcal{B}_c(\mathbb{R}^d)\ \ |\mu(\Lambda)| < \infty.$$

We introduce a mapping from the Plato space $\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)$ to the space of vector-valued Radon measures:

$$\Pi(\mathbb{R}_0^d \times \mathbb{R}^d) \ni \gamma = \sum_{(v_x, x) \in \gamma} \delta_{(v_x, x)} \to \eta = \mathcal{R}\gamma = \sum_{x \in \tau(\gamma)} v_x \delta_x \in \mathcal{M}(\mathbb{R}^d),$$

where the image of this mapping is denoted as $\mathbb{K}(\mathbb{R}^d)$.

$$\mathbb{K}(\mathbb{R}^d) = \mathcal{R}(\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)) \subset \mathcal{M}(\mathbb{R}^d).$$

The topology of $\mathbb{K}(\mathbb{R}^d)$ is derived from the topology of the configuration space $\Gamma(\mathbb{R}_0^d \times \mathbb{R}^d)$.

In this framework, the points of $\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)$ are identified as ideas, while their images under the reflection mapping $\mathcal{R} : \Pi(\mathbb{R}_0^d \times \mathbb{R}^d) \to \mathbb{K}(\mathbb{R}^d)$ are interpreted as observed objects (i.e., the mapped representations of the ideas). The structure and properties of the object space $\mathbb{K}(\mathbb{R}^d)$ are entirely governed by the underlying Plato space $\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)$, which serves as the foundational framework for all further analyses of $\mathbb{K}(\mathbb{R}^d)$.

From a physical perspective, this construction models a system of particles in the space $\mathbb{R}^d$, where each spatial position $x \in \mathbb{R}^d$ is associated with a velocity (or mark) $v_x \in \mathbb{R}_0^d$. This representation encapsulates the particle system's spatial configuration and dynamic behavior.

In the context of the presented mathematical framework, the cited impossibility of particles having zero velocity stems from the structure of the configuration spaces and the properties of the vector-valued Radon measures. In particular, the system considers particles described by pairs (v_x, x), where $x \in \mathbb{R}^d$ represents the spatial position and

$v_x \in \mathbb{R}_0^d$ is the velocity, with $\mathbb{R}_0^d = \mathbb{R}^d \setminus \{0\}$. Accordingly, the exclusion of the zero vector in $\mathbb{R}_0^d$ ($v_x \neq 0$) means that particles must have a non-zero velocity, which is a natural restriction intrinsic into the definition of the phase space:

$$\Gamma(\mathbb{R}_0^d \times \mathbb{R}^d) = \Big\{\gamma = \{(v_x, x)\} \subset \mathbb{R}_0^d \times \mathbb{R}^d \mid \gamma \text{ is locally finite}\Big\}.$$

Indeed, this requirement implies that each particle configuration is composed of velocities that lie strictly outside the zero vector in $\mathbb{R}^d$. Moreover, we have to consider *Plato space constraints*. Indeed, the Plato space $\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)$ imposes a bounded total velocity condition within any compact subset $\Lambda \subset \mathbb{R}^d$:

$$V_\Lambda(\gamma) = \sum_{x \in \tau(\gamma) \cap \Lambda} |v_x| < \infty, \quad \forall \gamma \in \Pi(\mathbb{R}_0^d \times \mathbb{R}^d),$$

where $\tau(\gamma) = \{x \mid (v_x, x) \in \gamma\}$ denotes the spatial projection. The exclusion of zero velocities ensures that the velocity magnitudes $|v_x|$ are strictly positive, which is necessary to prevent trivial solutions where the total velocity could be identically zero. It is also worth mentioning that the configuration space is linked to the cone of vector-valued discrete Radon measures $\mathbb{K}(\mathbb{R}^d)$ through reflection mapping:

$$\mathcal{R} : \gamma \mapsto \eta = \sum_{x \in \tau(\gamma)} v_x \delta_x,$$

where $v_x \neq 0$. For η to be meaningful within this framework, each v_x must contribute a non-zero vector, maintaining the integrity of the vector-valued measure. Last, but not least, let us underline that the use of singular velocity distributions, such as measures of the form

$$\lambda(dv) = |v|^{-\alpha} e^{-|v|^\beta} dv, \quad \alpha \in [d, d+1),\ \beta > 0,$$

further reinforces the exclusion of zero velocities. These distributions exhibit singular behavior near $v = 0$, making the probability measure λ diverge or become undefined at $v = 0$. Consequently, the measure cannot assign finite mass to configurations that include zero velocities.

2.1.2. Poisson Measure in $\Gamma(\mathbb{R}_0^d \times \mathbb{R}^d)$

Poisson measures were initially introduced as examples of quasi-invariant ergodic measures in configuration spaces in [9]. These measures, denoted as γ_λ for a parameter $\lambda > 0$, can be described in the space of finite point configurations $\Xi = \{\xi \subset \mathbb{R}^d : \xi \text{ is a countable set of points}\}$. The measure γ_λ is associated with the Poisson point process, where the probability of observing a configuration ξ in a domain $A \subset \mathbb{R}^d$ is given by the following formula:

$$\gamma_\lambda(\xi) = \exp\left(-\lambda \int_{\mathbb{R}^d} 1_{\{x \in A\}}\, dx\right) \prod_{x \in \xi} \lambda\, dx.$$

The Poisson measure captures the randomness of particle placements in space, with λ representing the intensity (mean number of points per unit volume). The study of Poissonian white noise, a distribution in the space of test functions $C_c^\infty(\mathbb{R}^d)$, was further developed in works such as [10–13]. Poissonian white noise, $\mathcal{N}$, is a generalized random field characterized by the property that for any test function $f \in C_c^\infty(\mathbb{R}^d)$, the value of the field at f is defined by the following stochastic integral:

$$\langle \mathcal{N}, f \rangle = \sum_{x \in \xi} f(x),$$

where ξ is a Poisson point process. This distribution can be thought of as the limit of discrete sums over the points of the Poisson process, and it plays a central role in stochastic analysis, providing a mathematical framework for random fields with independent increments. The analysis of Poisson measures has evolved further with significant contributions in the fields of stochastic analysis and differential geometry in configuration spaces. One of the notable developments in this direction is the formulation of integration-by-parts formulas and Dirichlet forms. In particular, for a Poisson measure γ_λ, the integration-by-parts formula takes the following form:

$$\int_\Xi \nabla f(\xi) \cdot \nabla g(\xi)\, d\gamma_\lambda(\xi) = \lambda \int_\Xi f(\xi) \Delta g(\xi)\, d\gamma_\lambda(\xi),$$

where f and g are smooth test functions, and Δ is the Laplace–Beltrami operator associated with the configuration space. This formula provides a rigorous connection between the gradient of a test function and its Laplacian, which is crucial for understanding the behavior of functionals in Poisson configurations. Moreover, the concept of Dirichlet forms in Poisson spaces, introduced in [8], has become a cornerstone in the study of stochastic processes. A Dirichlet form, $\mathcal{E}$, associated with the Poisson measure is typically written as follows:

$$\mathcal{E}(f, g) = \frac{1}{2} \int_\Xi (\nabla f(\xi) \cdot \nabla g(\xi)) d\gamma_\lambda(\xi),$$

which measures the "energy" associated with the transitions between different configurations in the space. The existence of Dirichlet forms is crucial for understanding the regularity properties of functionals defined in configuration spaces and plays a pivotal role in the study of stochastic differential equations driven by Poisson measures. Further advancements in differential geometry in configuration spaces have led to a deeper understanding of the geometric structure of the spaces of point configurations, including formulating various notions of curvature and using these ideas in stochastic processes. These developments provide a mathematical framework for understanding the interactions between particles in systems modeled by Poisson processes, enabling the study of complex phenomena such as non-equilibrium statistical mechanics and the dynamics of interacting particle systems. The analysis of such systems benefits from the deep interplay between stochastic calculus, differential geometry, and statistical mechanics, leading to new insights into the behavior of large, interacting systems of particles. These foundational contributions, from the early work in [9] to more recent developments in [8], have established Poisson measures as an indispensable tool in the study of random configurations and their associated dynamics, particularly in the context of stochastic processes in configuration spaces and the geometry of random fields.

The classical ideal gas is described by a pair (v_x, x) where $x \in \mathbb{R}^d$ represents the position and $v_x \in \mathbb{R}^d_0 = \mathbb{R}^d \setminus \{0\}$ represents the velocity of a particle. The phase space of the gas is then given by $\mathbb{R}^d_0 \times \mathbb{R}^d$. Microscopic states of the gas are represented by phase space configurations, which are locally finite subsets of $\mathbb{R}^d_0 \times \mathbb{R}^d$. The space of such configurations is denoted by the following:

$$\Gamma(\mathbb{R}^d_0 \times \mathbb{R}^d) \ni \gamma = \{(v_x, x)\} \subset \mathbb{R}^d_0 \times \mathbb{R}^d.$$

A macroscopic state of the gas is a probability measure in $\Gamma(\mathbb{R}^d_0 \times \mathbb{R}^d)$. For an ideal gas, such a measure is a Poisson point measure. Let us take a Radon measure λ in $\mathbb{R}^d_0$ and a non-atomic Lebesgue measure m in $\mathbb{R}^d$. Consider an intensity measure $\lambda \otimes m$ in $\Gamma(\mathbb{R}^d_0 \times \mathbb{R}^d)$ and the Poisson measure $\pi_{\lambda \otimes m}$ in $\Gamma(\mathbb{R}^d_0 \times \mathbb{R}^d)$. The following Laplace transform,

$$\int_{\Gamma(\mathbb{R}^d_0 \times \mathbb{R}^d)} \exp\Big(\sum_{(v_x, x) \in \gamma} \psi(v_x, x)\Big) \pi_{\lambda \otimes m}(d\gamma) = \int_{\mathbb{R}^d_0 \times \mathbb{R}^d} (\exp \psi(v_x, x) - 1) \lambda(dv) m(dx),$$

may characterize that measure for any continuous ψ with bounded support. The support of this measure is essentially smaller and may be restricted to the following pinpointed configurations:

$$\gamma = \{(v_x, x)\},\ \tau(\gamma) = \{x \mid (v_x, x) \in \gamma\} \subset \mathbb{R}^d.$$

Note that the set $\tau(\gamma)$, in general, is a discrete subset of $\mathbb{R}^d$ but is not locally finite, i.e., it is not a configuration in $\mathbb{R}^d$. To have a physically relevant property of finite local energy, we need to assume an additional property of the measure λ. Namely, we consider measures with finite first moments:

$$\int\limits_{\mathbb{R}^d_0} |v|\lambda(dv) < \infty.$$

This situation is at the centre of our considerations. Note that in the case of finite measures, λ, the Poisson measure $\pi_{\lambda \otimes m}$ may be considered in the framework of marked point processes [14]. The latter allows for an easier technical analysis.

We are interested in defining the following mapping:

$$\gamma \mapsto \eta = \sum_{x \in \tau(\gamma)} v_x \delta_x$$

which is a transformation mapping a configuration γ to a discrete vector-valued Radon measure in $\mathbb{R}^d$. The space of such measures, which becomes random when γ is random, exhibits rich mathematical structures involving complex analysis and geometry elements.

The scenario most analogous to the classical case in statistical physics arises with probability measures λ, particularly when dealing with Maxwellian (Gaussian) distributions. In these cases, integrating concerning velocities yields a Poisson distribution π_m in the configuration space of positions $\Gamma(\mathbb{R}^d)$. However, this result is unattainable when the intensity measure satisfies $\lambda(\mathbb{R}^d) = \infty$. The distinction between these two cases is fundamental, as the properties of the velocity distribution play a critical role in determining the spatial structure of the model in the case of an infinite measure λ.

2.1.3. Measures in $\mathbb{K}(\mathbb{R}^d)$

The theory of random measures is fundamental to the modern studies of probability and point processes, as discussed in [15,16]. Marked point processes can be understood as random discrete measures assigning vector-valued marks to points. The cone of vector-valued discrete Radon measures $\mathbb{K}(\mathbb{R}^d)$ is the following set:

$$\mathbb{K}(\mathbb{R}^d) = \left\{ \eta = \sum_{x \in \tau(\eta)} v_x \delta_x \in \mathcal{M}(\mathbb{R}^d) \middle| x \in \mathbb{R}^d,\ v_x \in \mathbb{R}^d_0 \right\}.$$

Probability measures in the cone $\mathbb{K}(\mathbb{R}^d)$ are fundamental tools for modeling various phenomena in mathematical physics, biology, and the representation theory of infinite-dimensional groups.

Our objective is to construct such measures by utilizing the reflection map $\mathcal{R}$ applied to carefully selected measures in the Plato space $\Pi(\mathbb{R}^d_0 \times \mathbb{R}^d)$. To exemplify this method, we begin with elementary cases related to Poisson measures. Specifically, we first define a non-atomic Radon measure λ in $\mathbb{R}^d_0$ that satisfies the following conditions:

$$\lambda(\mathbb{R}^d_0) = \infty;$$

$$\forall n \in \mathbb{N} \int\limits_{\mathbb{R}^d_0} |v|^n \lambda(dv) < \infty.$$

As a concrete example, we take

$$\lambda(dv) = \frac{1}{|v|^d} e^{-|v|^2} dv,$$

which represents a modified version of the Maxwell distribution for velocities in statistical physics, incorporating a singularity at $v = 0$. A broader class of such measures can be expressed in the following form:

$$\lambda(dv) = \frac{1}{|v|^\alpha} e^{-|v|^\beta} dv,$$

where $\alpha \in [d, d+1)$ and $\beta > 0$. These generalizations allow for flexibility in modeling systems with different velocity distributions.

Let $m(dx)$ denote a non-atomic Lebesgue measure in $\mathbb{R}^d$ and define the following intensity measure,

$$\sigma(dv, dx) = \lambda(dv) m(dx),$$

in $\mathbb{R}_0^d \times \mathbb{R}^d$. This measure combines the velocity distribution λ with the spatial measure m.

For any $\Lambda \in \mathcal{B}_c(\mathbb{R}_0^d \times \mathbb{R}^d)$, we have the following disjoint decomposition:

$$\Gamma(\Lambda) = \bigcup_{n=0}^{\infty} \Gamma^n(\Lambda),$$

where $\Gamma(\Lambda) := \{\gamma \in \Gamma(\mathbb{R}_0^d \times \mathbb{R}^d) : \gamma \subset \Lambda\}$ is the set of all configurations supported in Λ and $\Gamma^n(\Lambda) := \{\gamma \in \Gamma(\Lambda) : |\gamma| = n\}$ is the set of n-point configurations. The Lebesgue–Poisson measure in $\Gamma(\Lambda)$ is given by the following:

$$\mathcal{L}_\sigma := \sum_{n=0}^{\infty} \frac{1}{n!} \sigma^{(n)},$$

where $\sigma^{(n)}$ is the symmetric product measure in $\Gamma^n(\Lambda)$, defined as $\sigma^{(n)} := (\lambda \otimes m)^{\otimes n} \circ sym_n^{-1}$. The Poisson measure in $\Gamma(\Lambda)$ is defined as $\pi_\sigma^\Lambda := e^{-\sigma(\Lambda)} \mathcal{L}_\sigma$.

There is a standard definition of a Poisson measure π_σ in $\Gamma(\mathbb{R}_0^d \times \mathbb{R}^d)$; therefore, we can proceed as follows. For any $\psi \in C_0(\mathbb{R}_0^d \times \mathbb{R}^d)$ (continuous functions with compact supports), we define

$$< \psi, \gamma >= \sum_{(v,x) \in \gamma} \psi(v, x), \;\; \gamma \in \Gamma(\mathbb{R}_0^d \times \mathbb{R}^d).$$

The Poisson measure π_σ is defined via its Laplace transform:

$$\int_{\Gamma(\mathbb{R}_0^d \times \mathbb{R}^d)} e^{<\psi, \gamma>} \pi_\sigma(d\gamma) = \exp \int_{\mathbb{R}_0^d \times \mathbb{R}^d} (e^{\psi(v,x)} - 1) \lambda(dv) m(dx).$$

As a consequence of this definition, we have

$$\int_{\Gamma(\mathbb{R}_0^d \times \mathbb{R}^d)} < \psi, \gamma > \pi_\sigma(d\gamma) = \int_{\mathbb{R}_0^d \times \mathbb{R}^d} \psi(v, x) \lambda(dv) m(dx)$$

and this relation can be extended to the case of any $\psi \in L^1(\lambda \otimes m)$.

Lemma 1. *The Poisson measure π_σ in concentrated in $\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)$, i.e.,*

$$\pi_\sigma(\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)) = 1.$$

Proof. Take any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and define

$$\psi(v,x) = |v|1_\Lambda(x).$$

Then,

$$\int\limits_{\Gamma(\mathbb{R}_0^d \times \mathbb{R}^d)} \sum_{(v,x)\in\gamma} |v|1_\Lambda(x)\pi_\sigma(d\gamma) = m(\Lambda) \int\limits_{\mathbb{R}_0^d} |v|\lambda(dv) < \infty.$$

This means that $V_\Lambda < \infty$ π_σ-a.s., i.e., $\pi_\sigma(\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)) = 1$. □

To obtain measures in $\mathbb{K}(\mathbb{R}^d)$, we can employ the *pushforward* of measures in $\Pi(R_0^d \times \mathbb{R}^d)$ through the mapping $\mathcal{R}$. One crucial step is demonstrating the measurable structures' compatibility in $\Pi(R_0^d \times \mathbb{R}^d)$ and $\mathbb{K}(\mathbb{R}^d)$. Having already established the connection between $\mathbb{K}(\mathbb{R}^d)$ and $\Pi(R_0^d \times \mathbb{R}^d)$, it is natural to explore further the relationship induced by the mapping $\mathcal{R}$. For instance, one could examine the relationship between the σ-algebras $\mathcal{B}(\Pi(R_0^d \times \mathbb{R}^d))$ and $\mathcal{B}(\mathbb{K}(\mathbb{R}^d))$.

Theorem 1. *The image σ-algebra of $\mathcal{B}(\Pi(R_0^d \times \mathbb{R}^d))$ under $\mathcal{R}$ and $\mathcal{B}(\mathbb{K}(\mathbb{R}^d))$ coincide, i.e.,*

$$\mathcal{B}(\mathbb{K}(\mathbb{R}^d)) = \Big\{\mathcal{R}(A \cap \Pi(R_0^d \times \mathbb{R}^d)) \mid A \in \mathcal{B}(\Gamma(R_0^d \times \mathbb{R}^d))\Big\}.$$

Proof. The key idea is to apply Kuratowski's theorem (Theorem V.2.4 in [17]) using the fact that $\Pi(R_0^d \times \mathbb{R}^d) \in \mathcal{B}(\Gamma(R_0^d \times \mathbb{R}^d))$ (with a proof similar to Lemma 2.2.3 in [6]) along with the result that the trace σ-algebra is a separable Borel space (Theorem V.2.2 in [17]). □

We denote μ_λ, the image measure in $\mathbb{K}(\mathbb{R}^d)$, under the reflection map corresponding to π_σ. For $h \in \mathbb{R}^d$ and $\phi \in C_0(\mathbb{R}^d)$, we introduce a function $L_{h,\phi} : \mathbb{K}(\mathbb{R}^d) \to \mathbb{R}$ via the following:

$$L_{h,\phi}(\eta) = < h \otimes \phi, \eta > = \int\limits_{\mathbb{R}^d} \phi(x) < h, \eta(dx) > = \sum_{x\in\tau(\eta)} \phi(x) < h, v_x > .$$

The definition of the Poisson measure implies that

$$\int\limits_{\mathbb{K}} e^{<h\otimes\phi,\eta>}\mu_\lambda(d\eta) = \exp\Big(\int\limits_{\mathbb{R}_0^d\times\mathbb{R}^d} (e^{<h,v>\phi(x)} - 1)\lambda(dv)m(dx)\Big).$$

We define the function

$$\Phi_\lambda^h(r) = \exp\Big(\int\limits_{\mathbb{R}_0^d} (e^{<h,v>r} - 1)\lambda(dv)\Big), \quad r \in \mathbb{R},$$

which, within the integral, can be rewritten as

$$\int\limits_{\mathbb{K}} e^{<h\otimes\phi,\eta>}\mu_\lambda(d\eta) = e^{\int\limits_{\mathbb{R}^d} \log(\Phi_\lambda^h(\phi(x)))m(dx)}.$$

The definition of the measure μ_λ, through its Laplace transform, can be understood in terms of the equation above.

2.2. Harmonic Analysis in $\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)$

The inherent nonlinearity of infinite-dimensional spaces presents significant challenges in directly analyzing the dynamics formulated in $\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)$. To address this, we propose reformulating the equations within the space of finite configurations, $\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)$, which is a proper subset of $\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)$. Applying the K-transform facilitates this reformulation.

Kondratiev and Kuna utilized a comparable methodology in [5] for conducting harmonic analysis in configuration spaces over Riemannian manifolds.

2.2.1. The *K*-Transform

In what follows, we will introduce auxiliary space connected to the Plato space $\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)$ via the *K*-transform. Moreover, we will show relations between functions in $\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)$ and $\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)$.

Definition 1. *The Plato space of finite configurations* $\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)$ *is defined as follows:*

$$\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d) := \{\gamma \in \Pi(\mathbb{R}_0^d \times \mathbb{R}^d) \mid |\gamma| < \infty\},$$

where $|\cdot|$ *denotes the number of elements in a set. Its topology is induced by the set* $\Gamma_0(\mathbb{R}_0^d \times \mathbb{R}^d)$.

The spaces $\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)$ and $\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)$ play completely different roles. As mentioned before, $\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)$ represents the space of ideas, whereas $\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)$ is regarded as a mathematical construct that exists alongside $\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)$. Moreover, these spaces are also topologically different.

We will introduce *n*-point configurations, which are used to decompose the space.

Definition 2.

1. *For* $n \in \mathbb{N}_0$, *the set of n-point configurations is defined as follows:*

$$\Pi_0^{(n)}(\mathbb{R}_0^d \times \mathbb{R}^d) := \Big\{\gamma \in \Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d) \colon |\gamma| = n\Big\}.$$

2. *For a set* $\Lambda \subset \mathbb{R}_0^d \times \mathbb{R}^d$, *the set of all configurations supported in* Λ *is defined as follows:*

$$\Pi_0(\Lambda) := \Big\{\gamma \in \Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d) \colon \gamma \subset \Lambda\Big\}.$$

3. *A Borel set* $A \subset \Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)$ *is called bounded if there exists compact* $\Lambda \subset \mathbb{R}_0^d \times \mathbb{R}^d$ *and* $N \in \mathbb{N}$ *such that*

$$A \subset \bigcup_{n=0}^{N} \Pi_0^{(n)}(\Lambda).$$

We denote the system of all such sets by $\mathcal{B}_b(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d))$.

We have the following decompositions:

$$\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d) = \bigsqcup_{n=0}^{\infty} \Pi_0^{(n)}(\mathbb{R}_0^d \times \mathbb{R}^d) = \bigcup_{\Lambda \in \mathcal{B}_c(\mathbb{R}_0^d \times \mathbb{R}^d)} \Pi_0(\Lambda),$$

where the first union is disjoint and $\mathcal{B}_c(\mathbb{R}_0^d \times \mathbb{R}^d)$ *denotes all Borel subsets of* $\mathbb{R}_0^d \times \mathbb{R}^d$ *with compact closure.*

To introduce the *K*-transform between $\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)$ and $\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)$, we first need to define well-defined classes of functions in which this transform can be applied. Additionally, we introduce a specific class of measures that can extend the *K*-transform to a wider range of functions. We use the notation $B^0(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d))$ to represent the set of all measurable functions, $G \colon \Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d) \to \mathbb{R}$.

Definition 3.

1. *A function,* $G \colon \Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d) \to \mathbb{R}$, *is said to be bounded with local support if there exists* $C > 0$ *and* $\Lambda \in \mathcal{B}_c(\mathbb{R}_0^d \times \mathbb{R}^d)$ *such that the following estimate holds for all* $\eta \in \mathbb{K}_0(\mathbb{R}^d)$*:*

$$|G(\gamma)| \leq C \mathbb{1}_{\Pi_0(\Lambda)}(\gamma). \tag{1}$$

It is important to note that this implies that $G(\gamma) = 0$ *if* $\gamma \cap \Lambda^c \neq \emptyset$. *We denote by* $B_{\mathrm{ls}}(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d))$ *all measurable functions,* $G\colon \Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d) \to \mathbb{R}$, *which are bounded with local support.*

2. *A function,* $G\colon \Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d) \to \mathbb{R}$, *is called bounded with bounded support if there exists* $\Lambda \in \mathcal{B}_c(\mathbb{R}_0^d \times \mathbb{R}^d)$, $N \in \mathbb{N}$ *and* $C > 0$ *such that*

$$|G(\gamma)| \leq C \mathbb{1}_{\Pi_0(\Lambda)}(\gamma) \mathbb{1}_{\{|\gamma| \leq N\}}(\gamma), \tag{2}$$

i.e., $G(\gamma) = 0$ *whenever* $|\gamma| > N$ *or* $\gamma \cap \Lambda^c \neq \emptyset$. *We denote the space of all such measurable functions by* $B_{\mathrm{bs}}(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d))$. *Evidently, it follows that we have* $B_{\mathrm{bs}}(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)) \subset B_{\mathrm{ls}}(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d))$.

3. *A measure,* ρ, *in* $\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)$ *is called locally finite if for any* $\Lambda \in \mathcal{B}_c(\mathbb{R}_0^d \times \mathbb{R}^d)$ *and for any* $m \in \mathbb{N}_0$ *the value of* $\rho(\Pi_0^{(m)}(\Lambda))$ *is finite. Similarly,* $\rho(A)$ *is finite for all bounded measurable sets,* $A \subset \Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)$. *The space of all locally finite measures in* $\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)$ *is denoted by* $\mathcal{M}_{\mathrm{lf}}(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d))$.

In the following, we define the *K*-transform and its main properties.

Definition 4 ([5]). *Let* $G \in B_{\mathrm{ls}}(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d))$. *The K-transform of G is the function* $KG\colon \Pi(\mathbb{R}_0^d \times \mathbb{R}^d) \to \mathbb{R}$ *defined by the following:*

$$(K_\Pi G)(\gamma) = (KG)(\gamma) := \sum_{\xi \Subset \gamma} G(\xi),$$

where the inclusion of $\xi \Subset \gamma$ *means that the sum is taken over all finite subsets of* γ.

To avoid any confusion, the dependence on Π is omitted. We can see that by the definition of $B_{\mathrm{ls}}(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d))$, the *K*-transform is well defined in such functions.

We recall some results similar to the results which can be found in the theory of homogeneous configuration spaces [5].

Proposition 1 ([5]).

1. *The K-transform maps functions from* $B_{\mathrm{ls}}(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d))$ *into cylinder functions* $\mathcal{F}L^0(\Pi(\mathbb{R}_0^d \times \mathbb{R}^d))$, *i.e., for* $G \in B_{\mathrm{ls}}(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d))$, *there exists* $\Lambda \in \mathcal{B}_c(\mathbb{R}_0^d \times \mathbb{R}^d)$ *such that*

$$(KG)(\gamma) = (KG)(\gamma \cap \Lambda),$$

for all $\gamma \in \Pi(\mathbb{R}_0^d \times \mathbb{R}^d)$.

2. *The K-transform maps* $B_{\mathrm{bs}}(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d))$ *to polynomially bounded functions, i.e., for* $G \in B_{\mathrm{bs}}(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d))$, *there exist* $\Lambda \in \mathcal{B}_c(\mathbb{R}_0^d \times \mathbb{R}^d)$, $N \in \mathbb{N}$ *and* $C > 0$ *such that*

$$|KG|(\gamma) \leq C(1 + |\gamma \cap \Lambda|)^N, \ \gamma \in \Pi(\mathbb{R}_0^d \times \mathbb{R}^d).$$

3. *The mapping* $K\colon B_{\mathrm{ls}}(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)) \to \mathcal{F}B^0(\Pi(\mathbb{R}_0^d \times \mathbb{R}^d))$ *is invertible with*

$$K^{-1}F(\gamma) = \sum_{\xi \subset \gamma} (-1)^{|\gamma \setminus \xi|} F(\xi), \ \gamma \in \Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d).$$

4. *K is linear and positivity-preserving.*
5. *If* $G \in B_{\mathrm{ls}}(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d))$ *and continuous, then KG is also continuous.*

Let us consider the following example of the *K*-transform of a (Lebesgue–Poisson) coherent state corresponding to the function f.

Example 1 ([5]). *For a function, $f \in C_0(\mathbb{R}_0^d \times \mathbb{R}^d)$, we define the coherent state or Lebesgue–Poisson exponent as follows:*

$$e_{\mathcal{L}}(f)\colon \Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d) \to \mathbb{R},\ \gamma \mapsto e_\lambda(f,\gamma) := \prod_{(v,x)\in\gamma} f(v,x).$$

Then, $e_{\mathcal{L}}(f) \in B_{\mathrm{ls}}(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d))$. Its K-transform is given by the following:

$$(Ke_{\mathcal{L}}(f))(\gamma) = \prod_{(v,x)\in\gamma} (1 + f(v,x)),\ \gamma \in \Pi(\mathbb{R}_0^d \times \mathbb{R}^d).$$

We introduce the $\star$-convolution related to the K-transform as the standard convolution in $\mathbb{R}^d$ to the Fourier transform.

Definition 5. *Let $G_1, G_2 \in B_{\mathrm{ls}}(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d))$. We define the $\star$-convolution as follows:*

$$(G_1 \star G_2)(\gamma) := \sum_{(\xi_1,\xi_2,\xi_3)\in\mathcal{P}^3_{\emptyset}(\gamma)} G_1(\xi_1 \cup \xi_2) G_2(\xi_2 \cup \xi_3),\ \gamma \in \Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d),$$

where $\mathcal{P}^3_{\emptyset}(\gamma)$ denotes all partitions of γ into three parts, where the parts may be empty.

The following relation holds:

Proposition 2 ([5]). *Let $G_1, G_2 \in B_{\mathrm{ls}}(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d))$ be given. Then,*

$$K(G_1 \star G_2) = KG_1 \cdot KG_2.$$

2.2.2. Correlation Measures in $\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)$

Our objective is to establish categories of measures in $\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)$ that correspond to probability measures in $\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)$, utilized to model the state of our system. This approach is based on [2,5]. Additionally, we will demonstrate that the group of measures in $\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)$ with limited local moments allows us to extend the K-transform to L^1-spaces. To define a measure in $\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)$, we first introduce the pre-kernel $\mathcal{K}$.

Definition 6. *We define the following pre-kernel based on the K-transform by the following:*

$$\begin{aligned} &\mathcal{K}\colon \mathcal{B}_b(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)) \times \Pi(\mathbb{R}_0^d \times \mathbb{R}^d) \to [0,\infty) \\ &(A,\gamma) \mapsto \mathcal{K}(A,\gamma) := (K\mathbb{1}_A)(\gamma). \end{aligned} \tag{3}$$

We show that $\mathcal{K}$ is in fact a pre-kernel. The property $\mathcal{K}(\emptyset,\gamma) = 0$ for any $\gamma \in \Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)$ is obvious. For σ-additivity, let $A_i \in \mathcal{B}_b(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d))$, $i \in \mathbb{N}$ be disjoint sets such that their countable union also belongs to $\mathcal{B}_b(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d))$. Then, there exist $N \in \mathbb{N}$ and $\Lambda \in \mathcal{B}_c(\mathbb{R}_0^d \times \mathbb{R}^d)$ such that

$$\bigcup_{i=1}^{\infty} A_i \subset \bigcup_{k=0}^{N} \Pi_0^{(k)}(\Lambda).$$

This implies that for $\gamma \in \Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)$,

$$\mathcal{K}\left(\bigcup_{i=1}^{\infty} A_i, \gamma\right) = \sum_{\xi\Subset\gamma} \sum_{i=1}^{\infty} \mathbb{1}_{A_i}(\xi) = \sum_{\substack{\xi\Subset\gamma \\ |\gamma|\le N}} \sum_{i=1}^{\infty} \mathbb{1}_{A_i}(\xi) \sum_{i=1}^{\infty} \sum_{\substack{\xi\Subset\gamma \\ |\xi|\le N}} \mathbb{1}_{A_i}(\xi) = \sum_{i=1}^{\infty} \mathcal{K}(A,\gamma),$$

which completes the proof of the claim. Moreover, $\mathcal{K}$ can indeed be extended.

Lemma 2. *The pre-kernel $\mathcal{K}$ has a unique extension to a kernel in $\mathcal{B}(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)) \times \Pi(\mathbb{R}_0^d \times \mathbb{R}^d)$.*

Proof. Since $\mathcal{B}_b(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d))$ is a ring, it suffices to demonstrate the σ-finiteness of $\mathcal{K}(\cdot, \gamma)$ in order to obtain a unique extension to $\mathcal{B}(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d))$. For $A \in \mathcal{B}_b(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d))$, the sum

$$\mathcal{K}(A, \gamma) = \sum_{\xi \in \gamma} \mathbb{1}_A(\xi)$$

is finite. Therefore, by Carathéodory's theorem, $\mathcal{K}$ can be uniquely extended to a kernel in $\mathcal{B}(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)) \times \Pi(\mathbb{R}_0^d \times \mathbb{R}^d)$. □

We can further extend Relation (3) to more general functions.

Proposition 3. *Let $G\colon \Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d) \to \mathbb{R}$ be a measurable function with $G \geq 0$ or $G \in B_{\mathrm{ls}}(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d))$. Then,*

$$\int\limits_{\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)} G(\xi)\mathcal{K}(d\xi, \gamma) = \sum_{\xi \in \gamma} G(\xi) = (KG)(\gamma).$$

Proof. The function G can be approximated by a sequence of simple functions, namely the following:

$$G(\gamma) = \sum_{k=1}^{\infty} a_k \mathbb{1}_{A_k}(\gamma),$$

where $a_k \in \mathbb{R}$, $A \in \mathcal{B}_b(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d))$, $\gamma \in \Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)$. The identity can then be derived by taking monotone limits. For further details, see [2,5]. □

We can now construct measures in $\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)$ that correspond to probability measures in $\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)$ using the kernel $\mathcal{K}$.

Definition 7. *Let μ be a probability measure in $(\Pi(\mathbb{R}_0^d \times \mathbb{R}^d), \mathcal{B}(\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)))$. The corresponding correlation measure is defined in $(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d), \mathcal{B}(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)))$ by the following relation:*

$$\rho_\mu(A) := \int\limits_{\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)} \mathcal{K}(A, \gamma)\mu(d\gamma).$$

Locally finite correlation measures can be defined through their corresponding probability measures in $\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)$. However, to guarantee that the correlation measure is indeed locally finite, additional assumptions must be imposed on the measure μ.

Proposition 4. *Let μ be a probability measure in $(\Pi(\mathbb{R}_0^d \times \mathbb{R}^d), \mathcal{B}(\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)))$. Then, the corresponding correlation measure ρ_μ is locally finite if and only if the following holds: for any $\Lambda \in \mathcal{B}_c(\mathbb{R}_0^d \times \mathbb{R}^d)$ and $N \in \mathbb{N}$,*

$$\int\limits_{\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)} |\gamma \cap \Lambda|^N \mu(d\gamma) < \infty. \tag{4}$$

Definition 8. *A measure, μ, that satisfies Proposition (4) is said to have finite local moments of all orders. The space of all such measures is denoted by $\mathcal{M}^1_{\mathrm{fm}}(\Pi(\mathbb{R}_0^d \times \mathbb{R}^d))$.*

Proof of Proposition 4. The proof works analogously to the case of classical configuration space; see [5].

($\Leftarrow$) For any $A \in \mathcal{B}_b(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d))$, there exists $\Lambda \in \mathcal{B}_c(\mathbb{R}_0^d \times \mathbb{R}^d)$ and $N \in \mathbb{N}$ such that $A \subset \bigcup_{k=0}^N \Pi_0^{(k)}(\Lambda)$. Then,

$$\begin{aligned}\rho_\mu(A) &= \int_{\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)} \mathcal{K}(A,\gamma)\mu(d\gamma) \leq \sum_{k=0}^N \int_{\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)} \mathcal{K}(\Pi_0^{(k)}(\Lambda),\gamma)\mu(d\gamma) \\ &= \sum_{k=0}^N \int_{\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)} \binom{|\gamma \cap \Lambda|}{k} \mu(d\gamma),\end{aligned} \tag{5}$$

where the last equality is obtained using

$$\mathcal{K}(\Pi_0^{(k)}(\Lambda),\gamma) = \sum_{\xi \Subset \gamma} \mathbb{1}_{\Pi_0^{(k)}(\Lambda)}(\xi) = \sum_{\substack{\xi \Subset \gamma \\ |\gamma|=k}} \mathbb{1}_{\Pi_0^{(k)}(\Lambda)}(\xi) = \binom{|\gamma \cap \Lambda|}{k}.$$

As long as $\sum_{k=0}^N \int_{\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)} \binom{|\gamma \cap \Lambda|}{k} \mu(d\gamma)$ consists of a finite linear combination of powers of $|\gamma \cap \Lambda|$, the proof is concluded.
($\Rightarrow$) Conversely, since $|\gamma \cap \Lambda|^N$ can be written as a finite linear combination of terms of the form $\binom{|\gamma \cap \Lambda|}{k}$ for $k \leq N$, the integral is finite by a similar argument. □

For the class of measures $\mathcal{M}^1_{\text{fm}}(\Pi(\mathbb{R}_0^d \times \mathbb{R}^d))$, we can extend the K-transform to L^1-spaces related to these measures.

Proposition 5 ([5]). *Let $\mu \in \mathcal{M}^1_{\text{fm}}(\Pi(\mathbb{R}_0^d \times \mathbb{R}^d))$ be given. For all functions $G \in B_{\text{bs}}(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d))$, we have $G \in L^1(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d), \rho_\mu)$. Furthermore, if $G \geq 0$ or $G \in B_{\text{bs}}(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d))$, then*

$$\int_{\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)} G(\xi)\rho_\mu(d\xi) = \int_{\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)} (KG)(\gamma)\mu(d\gamma). \tag{6}$$

Proof. The proof follows directly, as in [5]. Since $\mu(\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)) = 1$, the restriction from $\Gamma(\mathbb{R}_0^d \times \mathbb{R}^d)$ to $\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)$ does not affect the identity.

There exists $C > 0$ and a set $A \in B_b(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d))$ such that $G|_{\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d) \setminus A} = 0$ and $|G(\xi)| \leq C$ for all $\xi \in \Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)$. By Proposition (4), we then obtain the following:

$$\int_{\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)} |G(\xi)|\rho_\mu(d\xi) \leq C\rho_\mu(A) < \infty.$$

The remainder of the proof follows using standard measure-theoretic methods.
□

Remark 1. *For a measure, $\mu \in \mathcal{M}^1_{\text{fm}}(\Pi(\mathbb{R}_0^d \times \mathbb{R}^d))$, we may define the correlation measure without using the kernel $\mathcal{K}$ directly via the following:*

$$\rho_\mu(A) := \int_{\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)} K\mathbb{1}_A(\gamma)\mu(d\gamma),\ A \in \mathcal{B}_b(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)).$$

This follows from Proposition 4, since $K\mathbb{1}_A \in L^1(\mu)$ for $A \in \mathcal{B}_b(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d))$.

Definition 9. *The remark above enables us to define the dual operator of K, i.e.,*

$$K\colon \mathcal{M}^1_{\mathrm{fm}}(\Pi(\mathbb{R}^d_0 \times \mathbb{R}^d)) \to \mathcal{M}_{\mathrm{lf}}(\Pi_0(\mathbb{R}^d_0 \times \mathbb{R}^d))$$
$$\mu \mapsto K^\mu := \rho_\mu.$$

To complete the extension of the K-transform, we require one final continuity result for this mapping.

Lemma 3 ([5]). *Let $\{G_n\}_{n\in\mathbb{N}} \subset B_{\mathrm{bs}}(\Pi_0(\mathbb{R}^d_0 \times \mathbb{R}^d))$ be a sequence which converges in $L^1(\Pi_0(\mathbb{R}^d_0 \times \mathbb{R}^d), \rho_\mu)$ for some measure $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Pi(\mathbb{R}^d_0 \times \mathbb{R}^d))$. Then, $\{KG_n\}_{n\in\mathbb{N}}$ converges in $L^1(\Pi(\mathbb{R}^d_0 \times \mathbb{R}^d), \mu)$.*

Proof. The proof follows by applying the triangle inequality: $|KG| \le K|G|$. □

We can now prove the extension result for the K-transform in L^1-spaces.

Theorem 2 ([5]). *Let $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Pi(\mathbb{R}^d_0 \times \mathbb{R}^d))$ be given. For any $G \in L^1(\Pi_0(\mathbb{R}^d_0 \times \mathbb{R}^d), \rho_\mu)$, we define*

$$KG(\gamma) := \sum_{\xi\in\gamma} G(\xi),$$

where the series almost surely converges absolutely to μ. Furthermore, we have the following estimate:

$$\|KG\|_{L^1(\mu)} \le \|K|G|\|_{L^1(\mu)} = \|G\|_{L^1(\rho_\mu)},$$

which implies that $KG \in L^1(\mu)$, and for all $G \in L^1(\rho_\mu)$,

$$\int_{\Pi_0(\mathbb{R}^d_0\times\mathbb{R}^d)} G(\xi)\rho_\mu(d\xi) = \int_{\Pi(\mathbb{R}^d_0\times\mathbb{R}^d)} (KG)(\gamma)\mu(d\gamma). \tag{7}$$

Proof. Let G be a positive measurable function in $\Pi_0(\mathbb{R}^d_0 \times \mathbb{R}^d)$ that is integrable with respect to ρ_μ. Then, there exists an approximating sequence, $\{G_n\}_{n\in\mathbb{N}} \subset B_{\mathrm{bs}}(\Pi_0(\mathbb{R}^d_0 \times \mathbb{R}^d))$, that converges pointwise to G monotonically. By the monotonicity of the sequence, we have

$$\lim_{n\to\infty} \sum_{\xi\in\gamma} G_n(\xi) = \sum_{\xi\in\gamma} G(\xi)$$

and by the monotone convergence, it follows that

$$\begin{aligned}\int_{\Pi(\mathbb{R}^d_0\times\mathbb{R}^d)} \sum_{\xi\in\gamma} G(\xi)\mu(d\gamma) &= \lim_{n\to\infty} \int_{\Pi(\mathbb{R}^d_0\times\mathbb{R}^d)} \sum_{\xi\in\gamma} G_n(\xi)\mu(d\gamma) \\ &= \lim_{n\to\infty} \int_{\Pi(\mathbb{R}^d_0\times\mathbb{R}^d)} KG_n(\xi)\mu(d\gamma).\end{aligned} \tag{8}$$

Using Fatou's lemma, we further obtain

$$\lim_{n\to\infty} \int_{\Pi_0(\mathbb{R}^d_0\times\mathbb{R}^d)} G_n(\xi)\rho_\mu(d\xi) = \int_{\Pi_0(\mathbb{R}^d_0\times\mathbb{R}^d)} G(\xi)\rho_\mu(d\xi).$$

Since $G \in L^1(\Pi_0(\mathbb{R}^d_0 \times \mathbb{R}^d), \rho_\mu)$, we conclude that

$$\int_{\Pi(\mathbb{R}^d_0\times\mathbb{R}^d)} \sum_{\xi\in\gamma} G(\xi)\mu(d\gamma) = \int_{\Pi_0(\mathbb{R}^d_0\times\mathbb{R}^d)} G(\xi)\rho_\mu(d\xi) < \infty.$$

For a general G, the proof is completed by decomposing the function into its positive and negative parts. □

The identity given by Proposition 2 can be extended, considering some conditions.

Proposition 6 ([5]). *Let $G_1, G_2 \in B_{\mathrm{ls}}(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d))$ and let $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Pi(\mathbb{R}_0^d \times \mathbb{R}^d))$. Then, the following identity,*

$$K(G_1 \star G_2) = KG_1 \cdot KG_2,$$

holds for μ almost surely if one of the following conditions is satisfied:

1. $G_1, G_2 \geq 0$;
2. $|G_1| \star |G_2| \in L^1(\rho_\mu)$ *(and consequentially $K(G_1 \star G_2) \in L^1(\mu)$);*
3. $G_1, G_2 \in L^1(\rho_\mu)$.

Proof. A direct consequence of Theorem 2. □

2.2.3. Correlation Functions in $\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)$

Our goal is to establish the existence of a correlation function associated with a specific subset of measures, denoted by $\mathcal{M}^1_{\mathrm{fm}}(\Pi(\mathbb{R}_0^d \times \mathbb{R}^d))$. This correlation function acts as the density function for the corresponding correlation measure ρ_μ. Such functions are of particular mathematical interest due to their widespread use in applications where they characterize the behavior of the system under consideration.

Definition 10. *A measure, $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Pi(\mathbb{R}_0^d \times \mathbb{R}^d))$, is locally absolutely continuous with respect to the Poisson measure π_σ if the measure μ^Λ is absolutely continuous with respect to π_σ^Λ for all $\Lambda \in \mathcal{B}_c(\mathbb{R}_0^d \times \mathbb{R}^d)$, where $\mu^\Lambda := \mu \circ p_\Lambda^{-1}$.*

Proposition 7 ([5]). *For a measure $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Pi(\mathbb{R}_0^d \times \mathbb{R}^d))$ which is locally absolutely continuous with respect to π_σ, the correlation measure ρ_μ is absolutely continuous with respect to the Lebesgue–Poisson measure $\mathcal{L}_\sigma$ introduced in Section 2.1. The density function has the following representation for any $\gamma \in \Pi_0(\Lambda)$:*

$$k_\mu(\gamma) = \frac{d\rho_\mu}{d\mathcal{L}_\sigma}(\gamma) = \int\limits_{\Pi(\Lambda)} \frac{d\mu^\Lambda}{d\pi_\sigma^\Lambda}(\gamma \cup \xi)\pi_\sigma^\Lambda(d\xi).$$

Definition 11. *The function $k_\mu \colon \Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d) \to \mathbb{R}$ defined by the previous proposition is called the correlation function corresponding to μ. Moreover, we have the decomposition $k_\mu \simeq \{k_\mu^{(n)}\}_{n=0}^\infty$, where for any $n \in \mathbb{N}$, $k_\mu^{(n)} \colon (\mathbb{R}_0^d \times \mathbb{R}^d)^n \to \mathbb{R}$ is a symmetric function with*

$$k_\mu^{(n)}(v_1, x_1, \ldots, v_n, x_n) := \begin{cases} k_\mu(\{(v_1, x_1), \ldots, (v_n, x_n)\}), & \text{if } |\{(v_1, x_1), \ldots, (v_n, x_n)\}| = n, \\ 0, & \text{otherwise.} \end{cases}$$

The functions $k_\mu^{(n)}$ are called n-point correlation functions.

In what follows, we present the Bogoliubov functional, which we will employ to define correlation functions in the cone $\mathbb{K}(\mathbb{R}^d)$. Bogoliubov originally developed this category of functionals [18] to establish correlation functions for systems in statistical mechanics. The Bogoliubov functional approach also investigates continuum interacting particle systems in [19].

Definition 12. *Let $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Pi(\mathbb{R}^d_0 \times \mathbb{R}^d))$. The Bogoliubov functional L^{Π}_{μ} corresponding to μ is a functional defined at each measurable function $\varphi\colon \mathbb{R}^d_0 \times \mathbb{R}^d \to \mathbb{R}$ by*

$$L^{\Pi}_{\mu}(\varphi) := \int\limits_{\Pi(\mathbb{R}^d_0 \times \mathbb{R}^d)} \prod_{(v,x)\in\gamma} (1 + \varphi(v,x))\mu(d\gamma),$$

provided that the right-hand side exists for $|\varphi|$.

The following proposition will be useful to define the correlation functions in $\mathbb{K}(\mathbb{R}^d)$ later.

Proposition 8 ([19])**.** *Under some assumptions, the Bogoliubov functional is the generating functional of the correlation function. In other words, for any $\varphi\colon \mathbb{R}^d_0 \times \mathbb{R}^d \to \mathbb{R}$ such that $L^{\Pi}_{\mu}(\varphi)$ is well defined, we have*

$$L^{\Pi}_{\mu}(\varphi) = \sum_{n=0}^{\infty} \frac{1}{n!} \int\limits_{(\mathbb{R}^d_0 \times \mathbb{R}^d)^n} \varphi(v_1,x_1)\cdots\varphi(v_n,x_n)\times \\ \times k^{(n)}_{\mu}(v_1,\ldots,x_n)\lambda(dv_1)m(dx_1)\ldots\lambda(dv_n)m(dx_n).$$

2.3. Harmonic Analysis in $\mathbb{K}(\mathbb{R}^d)$

We aim to present harmonic analysis in the cone of vector-valued discrete Radon measures in $\mathbb{R}^d$, denoted by $\mathbb{K}(\mathbb{R}^d)$, after introducing harmonic analysis in the Plato space $\Pi(\mathbb{R}^d_0 \times \mathbb{R}^d)$. To establish a relationship between $\Pi(\mathbb{R}^d_0 \times \mathbb{R}^d)$ and $\mathbb{K}(\mathbb{R}^d)$, we employ the reflection mapping $\mathcal{R}$.

2.3.1. The *K*-Transform

Initially, we focus on the space $\mathbb{K}_0(\mathbb{R}^d)$ and, similarly to the previous section, introduce subspaces of $\mathbb{K}_0(\mathbb{R}^d)$ to decompose the space. Subsequently, we proceed to introduce classes of functions in these spaces.

Definition 13.

1. *The set of discrete Radon measures with finite support is defined as follows:*
$$\mathbb{K}_0(\mathbb{R}^d) := \Big\{\eta \in \mathbb{K}(\mathbb{R}^d)\colon |\tau(\eta)| < \infty\Big\}.$$
2. *For $n \in \mathbb{N}_0$, the set of n-points measures is defined as follows:*
$$\mathbb{K}^{(n)}_0(\mathbb{R}^d) := \Big\{\eta \in \mathbb{K}_0(\mathbb{R}^d)\colon |\tau(\eta)| = n\Big\},\ n \in \mathbb{N}$$
and $\mathbb{K}^{(0)}_0(\mathbb{R}^d) = \{0\}$ the set consisting of the zero measure.
3. *For a compact set $\Lambda \subset \mathbb{R}^d$, the set of all measures supported in Λ is defined as follows:*
$$\mathbb{K}_0(\Lambda) := \Big\{\eta \in \mathbb{K}_0(\mathbb{R}^d)\colon \tau(\eta) \subset \Lambda\Big\}.$$
4. *A set $A \subset \mathbb{K}_0(\mathbb{R}^d)$ is called bounded if there exists a compact set $\Lambda \subset \mathbb{R}^d$ and $N \in \mathbb{N}$ such that*
$$A \subset \bigcup_{n=0}^{N} \mathbb{K}^{(n)}_0(\Lambda).$$
We denote the collection of all bounded Borel subsets of $\mathbb{K}_0(\mathbb{R}^d)$ by $\mathcal{B}_b(\mathbb{K}_0(\mathbb{R}^d))$.
5. *A bounded set $A \subset \mathbb{K}_0(\mathbb{R}^d)$ is said to have compact velocities if, additionally, there exists a compact set $I \subset \mathbb{R}^d_0$ such that*
$$A \cap \{\eta \in \mathbb{K}_0(\mathbb{R}^d) \mid \exists x \in \tau(\eta)\colon v_x \notin I\} = \emptyset.$$

We denote the collection of all such sets by $\mathcal{B}_{\mathrm{cm}}(\mathbb{K}_0(\mathbb{R}^d))$.

Note that we have

$$\mathbb{K}_0(\mathbb{R}^d) = \bigsqcup_{n=0}^{\infty} \mathbb{K}_0^{(n)}(\mathbb{R}^d)$$

and

$$\mathbb{K}_0(\mathbb{R}^d) = \bigcup_{\Lambda \in \mathcal{B}_c(\mathbb{R}^d)} \mathbb{K}_0(\Lambda),$$

where the first union is disjoint.

By the reflection mapping $\mathcal{R}$, we relate the subspaces of $\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)$ and $\mathbb{K}(\mathbb{R}^d)$.

Proposition 9. *The following relations hold:*

1. $\mathcal{R}\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d) = \mathbb{K}_0(\mathbb{R}^d)$.
2. $\mathcal{R}\Pi_0^{(n)}(\mathbb{R}_0^d \times \mathbb{R}^d) = \mathbb{K}_0^{(n)}(\mathbb{R}^d)$ *for any* $n \in \mathbb{N}_0$.
3. $\mathcal{R}\Pi_0(\mathbb{R}_0^d \times \Lambda) = \mathbb{K}_0(\Lambda)$ *for any set* $\Lambda \subset \mathbb{R}^d$.
4. *For any* $A \in \mathcal{B}_b(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d))$, *we have* $\mathcal{R}A \in \mathcal{B}_{\mathrm{cm}}(\mathbb{K}_0(\mathbb{R}^d))$ *and vice versa.*

Proof. We will now prove the first statement. The other statements follow similarly.

For $\gamma \in \Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)$, there is a representation, $\gamma = \sum_{i=1}^n \delta_{(v_i,x_i)}$. This implies that $\mathcal{R}\gamma = \sum_{i=1}^n v_i \delta_{x_i} \in \mathbb{K}_0(\mathbb{R}^d)$.

Let $\eta \in \mathbb{K}_0(\mathbb{R}^d)$. Again, we can represent it as $\eta = \sum_{i=1}^n v_i \delta_{x_i}$. By defining $\gamma = \sum_{i=1}^n \delta_{(v_i,x_i)}$, we obtain $\gamma \in \Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)$ and $\mathcal{R}\gamma = \eta$. □

We now introduce the corresponding function spaces in $\mathbb{K}_0(\mathbb{R}^d)$.

Definition 14.

1. *A function,* $G\colon \mathbb{K}_0(\mathbb{R}^d) \to \mathbb{R}$, *is said to be bounded with local support if there exist* $C > 0$ *and* $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ *such that the following estimate holds for all* $\eta \in \mathbb{K}_0(\mathbb{R}^d)$*:*

$$|G(\eta)| \leq C \mathbb{1}_{\mathbb{K}_0(\Lambda)}(\eta) \prod_{x \in \tau(\eta)} |v_x|. \tag{9}$$

   Note that this implies that $G(\eta) = 0$ *if* $\tau(\eta) \cap \Lambda^c \neq \emptyset$. *We denote by* $B_{\mathrm{ls}}(\mathbb{K}_0(\mathbb{R}^d))$ *all measurable functions,* $G\colon \mathbb{K}_0(\mathbb{R}^d) \to \mathbb{R}$, *which are bounded with local support.*
2. *A function,* $G\colon \mathbb{K}_0(\mathbb{R}^d) \to \mathbb{R}$, *is called bounded with bounded support if there exist* $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $N \in \mathbb{N}$, *and* $C > 0$ *such that*

$$|G(\eta)| \leq C \mathbb{1}_{\mathbb{K}_0(\Lambda)}(\eta) \mathbb{1}_{\{|\tau(\eta)| \leq N\}}(\eta),$$

   i.e., $G(\eta) = 0$ *whenever* $|\tau(\eta)| > N$ *or* $\tau(\eta) \cap \Lambda^c \neq \emptyset$. *We denote the space of all such functions by* $B_{\mathrm{bs}}(\mathbb{K}_0(\mathbb{R}^d))$.
3. *Taking into account the effect of the marks as above, we define a modified version of* $B_{\mathrm{bs}}(\mathbb{K}_0(\mathbb{R}^d))$. *For some* $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $N \in \mathbb{N}$ *and* $C > 0$, *we define the space* $\widetilde{B}_{\mathrm{bs}}(\mathbb{K}_0(\mathbb{R}^d))$ *as a space of all functions* $G\colon \mathbb{K}_0(\mathbb{R}^d) \to \mathbb{R}$ *which satisfy the bound*

$$|G(\eta)| \leq C \mathbb{1}_{\mathbb{K}_0(\Lambda)}(\eta) \mathbb{1}_{\{|\tau(\eta)| \leq N\}}(\eta) \prod_{x \in \tau(\eta)} |v_x|.$$

   Obviously, we have $\widetilde{B}_{\mathrm{bs}}(\mathbb{K}_0(\mathbb{R}^d)) \subset B_{\mathrm{ls}}(\mathbb{K}_0(\mathbb{R}^d))$.
4. *We define the space of bounded measurable functions with compact mark support as the set of all functions,* $G \in B_{\mathrm{bs}}(\mathbb{K}_0(\mathbb{R}^d))$, *such that there exists a compact set* $I \subset \mathbb{R}_0^d$ *for which*

$$|G(\eta)| \leq C \mathbb{1}_{\mathbb{K}_0(\Lambda)} \mathbb{1}_{\{|\tau(\eta)| \leq N\}} \prod_{x \in \tau(\eta)} \mathbb{1}_I(v_x), \tag{10}$$

where Λ, C, and N are as above. We denote the space of bounded functions with compact marks by $B_{\mathrm{cm}}(\mathbb{K}_0(\mathbb{R}^d))$.

5. *A measure, ρ on $\mathbb{K}_0(\mathbb{R}^d)$, is called locally finite if for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and for any $m \in \mathbb{N}_0$, the value of $\rho(\mathbb{K}_0^{(m)}(\Lambda))$ is finite. Equivalently, $\rho(A)$ is finite for all bounded measurable sets $A \subset \mathbb{K}_0(\mathbb{R}^d)$. The space of all locally finite measures in $\mathbb{K}_0(\mathbb{R}^d)$ is denoted by $\mathcal{M}_{\mathrm{lf}}(\mathbb{K}_0(\mathbb{R}^d))$.*
6. *A measure, ρ, in $\mathbb{K}_0(\mathbb{R}^d)$ is called mark-locally finite if $\rho(A) < \infty$ for all $A \in \mathcal{B}_{\mathrm{cm}}(\mathbb{K}_0(\mathbb{R}^d))$. A locally finite measure, ρ, is also mark-locally finite.*

We now establish the relationship between the function spaces in $\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)$ and $\mathbb{K}_0(\mathbb{R}^d)$. Using the reflection mapping $\mathcal{R}$, we map functions in $\mathcal{F}(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d))$, defined in $\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)$, to functions in $\mathcal{F}(\mathbb{K}_0(\mathbb{R}^d))$, defined in $\mathbb{K}_0(\mathbb{R}^d)$, as follows.

Definition 15. *We define the pushforward of functions in $\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)$ to $\mathbb{K}_0(\mathbb{R}^d)$ as follows:*

$$\begin{aligned} \mathcal{R}\colon \mathcal{F}(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)) &\to \mathcal{F}(\mathbb{K}_0(\mathbb{R}^d)) \\ F &\mapsto \mathcal{R}F := F \circ \mathcal{R}^{-1}, \end{aligned}$$

and, analogously, we may define the inverse mapping $\mathcal{R}^{-1}\colon \mathcal{F}(\mathbb{K}_0(\mathbb{R}^d)) \to \mathcal{F}(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d))$.

While for the function spaces in $\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)$ we require the compactness of mark support, for the function spaces in $\mathbb{K}_0(\mathbb{R}^d)$, according to the Definition 14, we have boundedness in the mark variables. The following proposition shows the relations between locally supported functions in $\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)$ and $\mathbb{K}_0(\mathbb{R}^d)$.

Proposition 10. *For the above spaces, the following relations hold:*

1. $\mathcal{R}B_{\mathrm{ls}}(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)) \not\subset B_{\mathrm{ls}}(\mathbb{K}_0(\mathbb{R}^d))$ *and* $B_{\mathrm{ls}}(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)) \not\supset \mathcal{R}^{-1}B_{\mathrm{ls}}(\mathbb{K}_0(\mathbb{R}^d))$;
2. $\mathcal{R}B_{\mathrm{bs}}(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)) = B_{\mathrm{cm}}(K_0(\mathbb{R}^d))$.

Proof.

1. Let $G \in B_{\mathrm{ls}}(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d))$ such that for some compact $A \subset \mathbb{R}_0^d$, we have $A \times \Lambda' \subset \Lambda$, where Λ is as it is in Definition 3 and $\Lambda' \subset \mathbb{R}^d$ are compact. We require the estimate

$$C\mathbb{1}_{\Pi_0(\Lambda)}(\mathcal{R}^{-1}\eta) \leq C_1 \mathbb{1}_{\mathbb{K}_0(\Lambda')}(\eta) \prod_{x\in\tau(\eta)} v_x$$

for some $C, C_1 > 0$. But, since A is as compact as possible, and the number of points in η is arbitrary, the right-hand side can be arbitrarily small. On the other hand, let $G \in B_{\mathrm{ls}}(\mathbb{K}_0(\mathbb{R}^d))$. To show $\mathcal{R}^{-1}G \in B_{\mathrm{ls}}(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d))$, we require

$$C\mathbb{1}_{\mathbb{K}_0(\Lambda')}(\mathcal{R}\gamma) \prod_{x\in\tau(\mathcal{R}\gamma)} v_x \leq C_1 \mathbb{1}_{\Pi_0(\Lambda)}(\mathcal{R}\gamma)$$

for some $C, C_1 > 0$ and Λ, Λ', as in the definitions above. Since there is no compactness requirement for the marks in $B_{\mathrm{ls}}(\mathbb{K}_0(\mathbb{R}^d))$, the left-hand side can be arbitrarily large.

2. Let $G \in B_{\mathrm{bs}}(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d))$. Then, there exist compact $\Lambda \in \mathcal{B}_c(\mathbb{R}_0^d \times \mathbb{R}^d)$; $N \in \mathbb{N}$ and $C > 0$ such that (2) holds. Then, there exists A such that $\Lambda \subset A \times \Lambda'$ for some $\Lambda' \in \mathcal{B}_c(\mathbb{R}^d)$. Then,

$$\begin{aligned} G(\mathcal{R}^{-1}\eta) &\leq C\mathbb{1}_{\Pi_0([A\times\Lambda')}(\mathcal{R}^{-1}\eta)\mathbb{1}_{\{|\mathcal{R}^{-1}\eta|\leq N\}}(\mathcal{R}^{-1}\eta) \\ &= C\mathbb{1}_{\mathbb{K}_0(\Lambda')}(\eta)\mathbb{1}_{\{|\tau(\eta)|\leq N\}}(\eta) \prod_{x\in\tau(\eta)} \mathbb{1}_A(v_x), \end{aligned}$$

which shows the first inclusion. On the other hand, let $G \in B_{\mathrm{cm}}(\mathbb{K}_0(\mathbb{R}^d))$. Then, there exist $\Lambda' \in \mathcal{B}_c(\mathbb{R}^d)$, $I \in \mathcal{B}_c(\mathbb{R}^d_0)$, $N \in \mathbb{N}$, and $C > 0$ such that (10) holds. Also, $I \times \Lambda' \in \mathcal{B}_c(\mathbb{R}^d_0 \times \mathbb{R}^d)$, and since

$$\mathbb{1}_{\mathbb{K}_0(\Lambda')}(\eta) \prod_{x \in \tau(\eta)} \mathbb{1}_I(v_x) = \mathbb{1}_{\Pi_0(I \times \Lambda')}(\mathcal{R}^{-1}\eta),$$

the claim follows.

□

We define the K-transform analogously to the case of the space $\Pi_0(\mathbb{R}^d_0 \times \mathbb{R}^d)$. We note that an estimate for the K-transform is obtained using the definition of function spaces (14).

Definition 16. *Let $G \in B_{\mathrm{ls}}(\mathbb{K}_0(\mathbb{R}^d))$. The K-transform of G is defined as the function $KG\colon \mathbb{K}(\mathbb{R}^d) \to \mathbb{R}$ of the following form:*

$$(K_{\mathbb{K}}G)(\eta) = (KG)(\eta) := \sum_{\xi \Subset \eta} G(\xi),$$

where the inclusion $\xi \Subset \eta$ is meant in the sense as in Definition 4.

Lemma 4. *For any $G \in B_{\mathrm{ls}}(\mathbb{K}_0(\mathbb{R}^d))$, the K-transform is well defined and the following estimate holds:*

$$|(KG)(\eta)| \leq C \prod_{x \in \tau(\eta) \cap \Lambda} (1 + |v_x|),$$

where C and Λ are as they are in Definition 14.

Proof. We have

$$|(KG)(\eta)| \leq \sum_{\xi \Subset \eta} |G(\eta)| \leq C \sum_{\substack{\xi \in \mathbb{K}_0(\Lambda) \\ \tau(\xi) \subset \tau(\eta)}} \prod_{x \in \tau(\xi)} |v_x| = C \prod_{x \in \tau(\eta) \cap \Lambda} (1 + |v_x|),$$

where the product in the last expression is finite if and only if the following sum is finite:

$$\sum_{x \in \tau(\eta) \cap \Lambda} |v_x|.$$

Since the latter holds by the definition of $\eta \in \mathbb{K}(\mathbb{R}^d)$, the claim follows. □

Similarly to the example in the previous section, we present the $\mathbb{K}_0(\mathbb{R}^d)$-analogue of coherent states.

Example 2. *For a function $\varphi \in C_0(\mathbb{R}^d)$ and a vector $h \in \mathbb{R}^d$, we define the coherent state as the function $e^h_{\mathbb{K}}(\varphi)\colon \mathbb{K}_0(\mathbb{R}^d) \to \mathbb{R}$ by the following:*

$$e_{\mathbb{K}}(\varphi, \eta) := \prod_{x \in \tau(\eta)} < h, v_x > \varphi(x),$$

Since φ is bounded, $e^h_{\mathbb{K}}(\varphi)$ fulfills bound (9). We can calculate its K-transform:

$$(Ke^h_{\mathbb{K}}(\varphi))(\eta) = \prod_{x \in \tau(\eta)} (1 + < h, v_x > \varphi(x)).$$

For the right-hand side to be well defined, the series $\sum_{x \in \tau(\eta)} < h, v_x > \varphi(x)$ needs to be convergent. This is given in our case since φ is compactly supported.

For $f^h_\varphi(v, x) := < h, v > \varphi(x)$, consider the Lebesgue–Poisson exponent $e_{\mathcal{L}}(f^h_\varphi)$ from Example 1. We see that

$$e^h_{\mathbb{K}}(\varphi, \mathcal{R}\gamma) = e_{\mathcal{L}}(f^h_\varphi, \gamma),\ \gamma \in \Pi_0(\mathbb{R}^d_0 \times \mathbb{R}^d).$$

We can relate the K-transform in $\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)$ and in $\mathbb{K}_0(\mathbb{R}^d)$ in the following way:

Proposition 11. *For $G \in B_{\mathrm{ls}}(\mathbb{K}_0(\mathbb{R}^d)) \cap \mathcal{R}B_{\mathrm{ls}}(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d))$ and $\eta \in \mathbb{K}(\mathbb{R}^d)$, the following holds:*

$$(K_{\mathbb{K}}G)(\eta) = (K_{\Pi}(\mathcal{R}^{-1}G))(\mathcal{R}^{-1}\eta).$$

Proof. Let $\eta = \sum_{i\in I} v_i\delta_{x_i}$, where $I \subset \mathbb{N}$. Then,

$$\begin{aligned}
(K_{\mathbb{K}}G)(\eta) &= \sum_{n=0}^{\infty} \sum_{\{i_1,\dots,i_n\}\subset I} G\left(\sum_{i=1}^{n} v_{i_k}\delta_{x_{i_k}}\right) = \sum_{n=0}^{\infty} \sum_{\{i_1,\dots,i_n\}\subset I} G\left(\mathcal{R}\left[\sum_{i=1}^{n} \delta_{(x_{i_k},v_{i_k})}\right]\right) \\
&= \sum_{n=0}^{\infty} \sum_{\{i_1,\dots,i_n\}\subset I} (\mathcal{R}^{-1}G)\left(\sum_{i=1}^{n} \delta_{(x_{i_k},v_{i_k})}\right) \\
&= \sum_{n=0}^{\infty} \sum_{\{i_1,\dots,i_n\}\subset I} (\mathcal{R}^{-1}G)\left(\mathcal{R}^{-1}\left[\sum_{i=1}^{n} v_{i_k}\delta_{x_{i_k}}\right]\right) \\
&= (K_{\Pi}(\mathcal{R}^{-1}G))(\mathcal{R}^{-1}\eta).
\end{aligned}$$

□

We need properties related to the K-transform, which will be used in the calculation below.

Lemma 5. *Let $G, G_1, G_2 \in B_{\mathrm{ls}}(\mathbb{K}_0(\mathbb{R}^d))$.*

1. *The K-transform has the following properties:*

$$\begin{aligned}
KG(\eta - v_x\delta_x) - KG(\eta) &= -(KG(\cdot + v_x\delta_x))(\eta - v_x\delta_x), \\
KG(\eta + v_x\delta_x) - KG(\eta) &= (KG(\cdot + v_x\delta_x))(\eta).
\end{aligned}$$

2. *The K-transform and the $\star$-convolution have the following relation:*

$$K(G_1 \star G_2) = KG_1 \cdot KG_2.$$

Proof. Proof can be conducted using combinatorial arguments [5]. □

The following lemma is needed for calculations in space of finite measures. It is also known as the Minlos Lemma.

Lemma 6 ([20])**.** *Let $\mathcal{L}_\sigma$ be the Lebesgue–Poisson measure in $\mathbb{K}_0(\mathbb{R}^d)$, associated with some intensity measure $\sigma = \lambda \otimes m$.*

1. *Let $G\colon \mathbb{K}_0(\mathbb{R}^d) \to \mathbb{R}$, $H\colon (\mathbb{K}_0(\mathbb{R}^d))^2 \to \mathbb{R}$. Then,*

$$\begin{aligned}
&\int_{\mathbb{K}_0(\mathbb{R}^d)} \int_{\mathbb{K}_0(\mathbb{R}^d)} G(\xi_1 + \xi_2)H(\xi_1, \xi_2)\mathcal{L}_\sigma(d\xi_1)\mathcal{L}_\sigma(d\xi_2) \\
&\qquad = \int_{\mathbb{K}_0(\mathbb{R}^d)} G(\eta) \sum_{\xi\subset\eta} H(\xi, \eta - \xi)\mathcal{L}_\sigma(d\eta).
\end{aligned}$$

2. *Let $H\colon \mathbb{K}_0(\mathbb{R}^d) \times \mathbb{R}_0^d \times \mathbb{R}^d \to \mathbb{R}$. Then,*

$$\begin{aligned}
&\int_{\mathbb{K}_0(\mathbb{R}^d)} \sum_{x\in\tau(\eta)} H(\eta, v_x, x)\mathcal{L}_\sigma(d\eta) \\
&\qquad = \int_{\mathbb{K}_0(\mathbb{R}^d)} \int_{\mathbb{R}_0^d\times\mathbb{R}^d} H(\eta + v\delta_x)\sigma(dv, dx)\mathcal{L}_\sigma(d\eta),
\end{aligned}$$

provided that at least one side of the equation exists.

We aim to extend the K-transform to the whole space, $L^1(\rho)$, for (mark-)locally finite measures ρ in $\mathbb{K}_0(\mathbb{R}^d)$. For this purpose, we show that $B_{\mathrm{cm}}(\mathbb{K}_0(\mathbb{R}^d))$ is dense in L^1-spaces. Similarly, we prove that $\widetilde{B}_{\mathrm{bs}}(\mathbb{K}_0(\mathbb{R}^d))$ is dense in L^1 with a modified measure.

By Proposition 10, we have $B_{\mathrm{cm}}(\mathbb{K}_0(\mathbb{R}^d)) \subset B_{\mathrm{ls}}(\mathbb{K}_0(\mathbb{R}^d)) \cap \mathcal{R}B_{\mathrm{ls}}(\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d))$. Hence, the relation in Proposition 11 holds for a set of functions in $L^1(\mathbb{K}_0, \rho)$ for a class of measures ρ.

Lemma 7. *For any locally finite measure, ρ, the space $B_{\mathrm{bs}}(\mathbb{K}_0(\mathbb{R}^d))$ is dense in $L^1(\rho)$.*

Proof. Let $G \in L^1(\mathbb{K}_0(\mathbb{R}^d), \rho)$ for some measure, ρ, in $\mathbb{K}_0(\mathbb{R}^d)$. We begin by approximating unbounded functions with bounded support. We define

$$G_n(\eta) := \Big[G(\eta)\mathbb{1}_{\mathbb{K}_0(B_n)}(\eta)\mathbb{1}_{\{|\tau(\eta)|\le n\}}(\eta)\Big] \wedge n$$
$$G'_n(\eta) := \Big[G(\eta)\mathbb{1}_{\mathbb{K}_0(B_n)}(\eta)\mathbb{1}_{\{|\tau(\eta)|\le n\}}(\eta)\Big],$$

where $B_n \subset \mathbb{R}^d$ is the ball with the radius n centered at 0. Then, $G_n \in B_{\mathrm{bs}}(\mathbb{K}_0(\mathbb{R}^d))$ and

$$\begin{aligned}\|G_n(\eta) - G'_n(\eta)\|_{L^1(\rho)} &= \int_{\mathbb{K}_0(\mathbb{R}^d)} |G_n(\eta) - G'_n(\eta)|\rho(d\eta)\\ &= \int_{\mathbb{K}_0(\mathbb{R}^d)} |G'_n(\eta)|\mathbb{1}_{\{|G_n(\eta)|\ge n\}}(\eta)\rho(d\eta)\\ &\le \int_{\mathbb{K}_0(\mathbb{R}^d)} |G(\eta)|\mathbb{1}_{\{|G(\eta)|\ge n\}}(\eta)\rho(d\eta).\end{aligned}$$

Since $G \in L^1(\mathbb{K}_0(\mathbb{R}^d), \rho)$, the last term converges to 0 for $n \to \infty$. Next, we define

$$G''_n(\eta) = G(\eta)\mathbb{1}_{\mathbb{K}_0(B_n)}(\eta).$$

We recall that $\mathbb{K}_0(\mathbb{R}^d)$ can be decomposed into disjoint n-point configurations, i.e.,

$$\mathbb{K}_0(\mathbb{R}^d) = \bigcup_{m=0}^{\infty} \mathbb{K}_0^{(m)}(\mathbb{R}^d),\ \mathbb{K}_0^{(m)}(\mathbb{R}^d) = \Big\{\eta \in \mathbb{K}_0(\mathbb{R}^d)\colon |\tau(\eta)| = m\Big\}.$$

Using this decomposition, we obtain

$$\begin{aligned}\|G'_n(\eta) - G''_n(\eta)\| &= \int_{\mathbb{K}_0(\mathbb{R}^d)} |G'_n(\eta) - G''_n(\eta)|\rho(d\eta)\\ &= \sum_{m=n+1}^{\infty} \int_{\mathbb{K}_0^{(n)}(\mathbb{R}^d)} |G''_n(\eta)|\rho(d\eta) \le \sum_{m=n+1}^{\infty} \int_{\mathbb{K}_0^{(n)}(\mathbb{R}^d)} |G(\eta)|\rho(d\eta)\end{aligned}$$

and since $G \in L^1(\mathbb{K}_0(\mathbb{R}^d))$, the series is absolutely convergent. Therefore, the last expression tends to 0 as $n \to \infty$. For the final step, observe that the increasing sequence $\{\mathbb{K}_0(B_n)\}_{n=1}^{\infty}$ approximates $\mathbb{K}_0(\mathbb{R}^d)$, and thus

$$\|G''_n - G\|_{L^1} = \int_{\mathbb{K}_0(\mathbb{R}^d)} |G(\eta)|\rho(d\eta) - \int_{\mathbb{K}_0(B_n)} |G(\eta)|\rho(d\eta),$$

which converges to 0 as $n \to \infty$ by the argument provided above. □

Corollary 1. *We define the following density function:*

$$f(\eta) = \prod_{x\in\tau(\eta)} \frac{1}{|v_x|}.$$

Then, the space $\widetilde{B}_{\mathrm{bs}}(\mathbb{K}_0(\mathbb{R}^d))$ *is dense in* $L^1(\mathbb{K}_0(\mathbb{R}^d), f\rho)$.

Proof. Let $G \in L^1(f\rho)$. By definition, we have

$$\|G\|_{L^1(f\rho)} = \int\limits_{\mathbb{K}_0(\mathbb{R}^d)} |G(\eta)| \prod_{x\in\tau(\eta)} \frac{1}{|v_x|}\rho(d\eta) < \infty,$$

which implies that $G \cdot f \in L^1(\rho)$. Since $B_{\mathrm{bs}}(\mathbb{K}_0(\mathbb{R}^d))$ is dense in $L^1(\rho)$, there exists a sequence $\{G_n\}_{n=1}^{\infty} \subset B_{\mathrm{bs}}(\mathbb{K}_0(\mathbb{R}^d))$ such that

$$\|G_n - G\cdot f\|_{L^1(\rho)} \to 0,\ n\to\infty.$$

On the other hand, the sequence $\{\widetilde{G}_n\}_{n=1}^{\infty}$ is in $\widetilde{B}_{\mathrm{bs}}(\mathbb{K}_0(\mathbb{R}^d))$, where $\widetilde{G}_n := \frac{G_n}{f}$ and satisfies

$$\widetilde{G}_n(\eta) = \frac{G_n(\eta)}{f(\eta)} \le C\mathbb{1}_{\mathbb{K}_0(\Lambda)}(\eta)\mathbb{1}_{\{|\tau(\eta)|\le N\}}(\eta) \prod_{x\in\tau(\eta)} v_x.$$

Moreover, $\widetilde{G}_n$ converges to G in $L^1(f\rho)$:

$$\begin{aligned}\|\widetilde{G}_n - G\|_{L^1(f\rho)} &= \int\limits_{\mathbb{K}_0(\mathbb{R}^d)} \left|\frac{G_n}{f} - G\right| f d\rho \\ &= \int\limits_{\mathbb{K}_0(\mathbb{R}^d)} |G_n - Gf| d\rho = \|G_n - Gf\|_{L^1(\rho)} \to 0\ n\to\infty.\end{aligned}$$

This completes the proof. □

The following is an example of a locally finite measure in $\mathbb{K}_0(\mathbb{R}^d)$, derived from the Lebesgue–Poisson measure in $\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)$.

Example 3. *Let* λ *be a locally finite measure in* $\mathbb{R}_0^d$ *and* m *a non-atomic measure in* $\mathbb{R}^d$ *(e.g., the Lebesgue measure). We define the measure* $\mathcal{L} = \mathcal{L}_{\lambda\otimes m}$ *as:*

$$\begin{aligned}&\int\limits_{\mathbb{K}_0(\mathbb{R}^d)} F(\eta)\mathcal{L}(d\eta) \\ &\quad = F(0) + \sum_{n=1}^{\infty} \frac{1}{n!} \int\limits_{(\mathbb{R}_0^d\times\mathbb{R}^d)^n} F\left(\sum_{i=1}^{n} v_i\delta_{x_i}\right)\lambda(dv_1)\dots\lambda(dv_n)m(dx_1)\dots m(dx_n),\end{aligned}$$

where $F\colon \mathbb{K}_0(\mathbb{R}^d) \to \mathbb{R}$ *such that the above expression exists and* 0 *denotes the zero measure. Then,* $\mathcal{L}$ *is locally finite.*

Proposition 12. *Let* $\rho \in \mathcal{M}_{\mathrm{lf}}(\mathbb{K}_0(\mathbb{R}^d))$. *The space* $B_{\mathrm{cm}}(\mathbb{K}_0(\mathbb{R}^d))$ *is dense in* $L^1(\rho)$ *as well as* $L^1(f\rho)$, *where* f *is the density function from Corollary 1.*

Proof. By Lemma 7 and Corollary 1, it is enough to show that functions in the spaces $B_{\mathrm{bs}}(\mathbb{K}_0(\mathbb{R}^d))$ and $\widetilde{B}_{\mathrm{bs}}(\mathbb{K}_0(\mathbb{R}^d))$ can be approximated by functions in $B_{\mathrm{cm}}(\mathbb{K}_0(\mathbb{R}^d))$, with

the convergence occurring with respect to $L^1(\rho)$ and $L^1(f\rho)$, respectively. Consider $G \in B_{\mathrm{bs}}(\mathbb{K}_0(\mathbb{R}^d))$. We define the sequence $\{G_n\}_{n=0}^{\infty}$ as follows:

$$G_n(\eta) := G(\eta) \cdot \prod_{x \in \tau(\eta)} \mathbb{1}_{[\frac{1}{n}, n]}(|v_x|),\ \eta \in \mathbb{K}_0(\mathbb{R}^d).$$

Then, we have

$$|G_n(\eta)| \leq C \mathbb{1}_{\mathbb{K}_0(\Lambda)}(\eta) \mathbb{1}_{\{|\tau(\eta)| \leq N\}} \prod_{x \in \tau(\eta)} \mathbb{1}_{[\frac{1}{n}, n]}(|v_x|),$$

where C, Λ and N are defined as in Definition 14. This shows that $G_n \in B_{\mathrm{cm}}(\mathbb{K}_0(\mathbb{R}^d))$. Next, we show that G_n approximates G as $n \to \infty$. Specifically,

$$\begin{aligned}
\|G_n - G\|_{L^1} &= \int_{\mathbb{K}_0(\mathbb{R}^d)} |G_n(\eta) - G(\eta)| \rho(d\eta) \\
&= \int_{\mathbb{K}_0(\mathbb{R}^d)} |G(\eta)| \left| 1 - \prod_{x \in \tau(\eta)} \mathbb{1}_{[\frac{1}{n}, n]}(|v_x|) \right| \rho(d\eta) \\
&= \int_{\mathbb{K}_0(\mathbb{R}^d)} |G(\eta)| \rho(d\eta) - \int_{\mathbb{K}_0(\mathbb{R}^d)} |G(\eta)| \prod_{x \in \tau(\eta)} \mathbb{1}_{[\frac{1}{n}, n]} |v_x| \rho(d\eta).
\end{aligned}$$

Since $G \in L^1(\rho)$, by Lebesgue's theorem, it suffices to show that $G_n \to G$ pointwisely. We fix $\eta \in \mathbb{K}_0(\mathbb{R}^d)$. Since $\tau(\eta)$ is finite, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $|v_x| \in [\frac{1}{n}, n]$ for every $x \in \tau(\eta)$. Therefore,

$$\prod_{x \in \tau(\eta)} \mathbb{1}_{[\frac{1}{n}, n]}(|v_x|) = 1\ \forall n \geq n_0,$$

which implies that $G_n(\eta) = G(\eta)$ for all $n \geq n_0$, i.e., $G_n \to G$ pointwisely. The above arguments establish that $G_n \to G$ in L^1, completing the proof.

The proof for the density of $B_{\mathrm{cm}}(\mathbb{K}_0(\mathbb{R}^d))$ in $L^1(f\rho)$ follows the same reasoning. Specifically, the estimate for G_n as above for $G \in B_{\mathrm{bs}}(\mathbb{K}_0(\mathbb{R}^d))$ is given by

$$\begin{aligned}
|G_n(\eta)| &\leq C \mathbb{1}_{\mathbb{K}_0(\Lambda)}(\eta) \mathbb{1}_{\{|\tau(\eta)| \leq N\}} \prod_{x \in \tau(\eta)} |v_x| \mathbb{1}_{[\frac{1}{n}, n]}(|v_x|) \\
&\leq C n^N \mathbb{1}_{\mathbb{K}_0(\Lambda)}(\eta) \mathbb{1}_{\{|\tau(\eta)| \leq N\}} \prod_{x \in \tau(\eta)} \mathbb{1}_{[\frac{1}{n}, n]}(|v_x|),
\end{aligned}$$

which also implies that $G_n \in B_{\mathrm{cm}}(\mathbb{K}_0(\mathbb{R}^d))$. □

2.3.2. Correlation Measures in $\mathbb{K}_0(\mathbb{R}^d)$

We aim at establishing correlation measures for $\mathbb{K}_0(\mathbb{R}^d)$ that correspond to probability measures μ in $(\mathbb{K}(\mathbb{R}^d), \mathcal{B}(\mathbb{K}(\mathbb{R}^d)))$. Additionally, we aim to expand the K-transform to L^1-type spaces for appropriate classes of measures. We will follow a similar approach to that used for $\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)$.

Given the unique structure of the spaces $\mathbb{K}(\mathbb{R}^d)$ and $\mathbb{K}_0(\mathbb{R}^d)$, we must consider the characteristics of measures concerning the marks.

Definition 17. *The pre-kernel $\mathcal{K}$ derived from the K-transform is defined as a map:*

$$\begin{aligned}
&\mathcal{K}\colon \mathcal{B}_b(\mathbb{K}_0(\mathbb{R}^d)) \times \mathbb{K}(\mathbb{R}^d) \to [0, \infty), \\
&(A, \eta) \mapsto \mathcal{K}(A, \eta) := (K \mathbb{1}_A)(\eta).
\end{aligned}$$

Similarly to the case of $\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)$, it can be shown that $\mathcal{K}$ is a pre-kernel and the same extension result holds:

Lemma 8. *The pre-kernel $\mathcal{K}$ can be uniquely extended to a kernel in $\mathcal{B}(\mathbb{K}_0(\mathbb{R}^d)) \times \mathbb{K}(\mathbb{R}^d)$.*

Proof. Similar to the proof of Lemma 2. □

Proposition 13. *Let $G\colon \mathbb{K}_0(\mathbb{R}^d) \to \mathbb{R}$ be a measurable function with $G \geq 0$ or $G \in B_{\mathrm{ls}}(\mathbb{K}_0(\mathbb{R}^d))$. Then,*

$$\int\limits_{\mathbb{K}_0(\mathbb{R}^d)} G(\xi)\mathcal{K}(d\xi, \eta) = \sum_{\xi \in \eta} G(\xi) = (KG)(\eta).$$

Proof. Similar to the proof of Proposition 3. □

Proposition 13 shows how the K-transform relates to the pre-kernel $\mathcal{K}$ for more general functions.

Now, we can use the kernel $\mathcal{K}$ to construct measures in $(\mathbb{K}_0(\mathbb{R}^d), \mathcal{B}(\mathbb{K}_0(\mathbb{R}^d)))$ corresponding to the probability measures in $(\mathbb{K}(\mathbb{R}^d), \mathcal{B}(\mathbb{K}(\mathbb{R}^d)))$.

Definition 18. *Let μ be a probability measure in the space $(\mathbb{K}(\mathbb{R}^d), \mathcal{B}(\mathbb{K}(\mathbb{R}^d)))$. The corresponding correlation measure is defined as a measure in $(\mathbb{K}_0(\mathbb{R}^d), \mathcal{B}(\mathbb{K}_0(\mathbb{R}^d)))$ by the following relation:*

$$\rho_\mu(A) := \int\limits_{\mathbb{K}(\mathbb{R}^d)} \mathcal{K}(A, \eta)\mu(d\eta).$$

In the preceding section, we focused on the group of locally finite correlation measures ρ_μ in $(\mathbb{K}_0(\mathbb{R}^d)$. Although it is common for such correlation measures to be mark-locally finite rather than locally finite in practical applications, studying the class of locally finite measures remains meaningful.

Proposition 14. *Let μ be a probability measure in $(\mathbb{K}(\mathbb{R}^d), \mathcal{B}(\mathbb{K}(\mathbb{R}^d)))$. Then, the corresponding correlation measure ρ_μ is locally finite if and only if the following holds: for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and $N \in \mathbb{N}$,*

$$\int\limits_{\mathbb{K}(\mathbb{R}^d)} |\tau(\eta) \cap \Lambda|^N \mu(d\eta) < \infty.$$

A measure, μ, with the above property is said to have finite local moments of all orders. The space of all such measures is denoted by $\mathcal{M}^1_{\mathrm{fm}}(\mathbb{K}(\mathbb{R}^d))$.

Proof. The proof works analogously to the one of Proposition 4. □

We introduce an example of measures in $\mathbb{K}_0(\mathbb{R}^d)$, which are not locally finite but at least mark-locally finite. We examine a certain type of measures, mentioned above. For a measure, ρ, in $\mathbb{K}_0(\mathbb{R}^d)$, we set

$$\widetilde{\rho}(d\eta) := f(\eta)\rho(d\eta),$$

where $f\colon \mathbb{K}_0(\mathbb{R}^d) \to (0, \infty)$ is the density function defined as follows:

$$f(\eta) = \prod_{x \in \tau(\eta)} \frac{1}{|v_x|}.$$

Lemma 9. *Let ρ be a locally finite measure. Then, $\widetilde{\rho}$ is mark-locally finite.*

Proof. Let $A \in \mathcal{B}_{\mathrm{cm}}(\mathbb{K}_0(\mathbb{R}^d))$. Then, $A \in \mathcal{B}_b(\mathbb{K}_0(\mathbb{R}^d))$. Furthermore, there exists $a > 0$ such that for all $\eta \in A$, we have $|v_x| \geq a$ for all $x \in \tau(\eta)$. Then,

$$\begin{aligned}\widetilde{\rho}(A) &= \int\limits_{\mathbb{K}_0(\mathbb{R}^d)} \mathbb{1}_A(\eta)\widetilde{\rho}(d\eta) = \int\limits_{\mathbb{K}_0(\mathbb{R}^d)} \mathbb{1}_A(\eta) f(\eta)\rho(d\eta) \\ &\leq \int\limits_{\mathbb{K}_0(\mathbb{R}^d)} \mathbb{1}_A(\eta)\max\left(1, \frac{1}{a^N}\right)\rho(d\eta) = \max\left(1, \frac{1}{a^N}\right)\rho(A) < \infty.\end{aligned}$$

□

2.3.3. Correlation Functions in $\mathbb{K}_0(\mathbb{R}^d)$

As noted earlier, correlation measures in applications are typically mark-locally finite but not locally finite. We aim to establish a density function specifically for this class of mark-locally finite correlation measures in $\mathbb{K}_0(\mathbb{R}^d)$. Consequently, our analysis will concentrate on this particular class of measures.

Definition 19. *Let μ be a probability measure in $(\mathbb{K}(\mathbb{R}^d), \mathcal{B}(\mathbb{K}(\mathbb{R}^d)))$.*

1. *Let $\Lambda \subset \mathbb{R}_0^d \times \mathbb{R}^d$. For $\eta \in \mathbb{K}(\mathbb{R}^d)$ of the form $\eta = \sum_{x\in\tau(\eta)} v_x\delta_x$, we define the projection with marks as follows:*
$$p_\Lambda(\eta) = \sum_{\substack{x\in\tau(\eta)\\(v_x,x)\in\Lambda}} v_x\delta_x.$$
The projection measure is defined as follows:
$$\mu^\Lambda := \mu \circ p_\Lambda^{-1}.$$
2. *The measure μ is called mark-locally absolutely continuous with respect to the measure μ_λ if for any compact $\Lambda \subset \mathbb{R}_0^d \times \mathbb{R}^d$, the measure μ^Λ is absolutely continuous with respect to μ_λ^Λ.*

We use a pullback correlation measure in the Plato space to examine mark-locally finite correlation measures in $\mathbb{K}_0(\mathbb{R}^d)$.

Proposition 15. *A correlation measure ρ_μ on $\mathbb{K}_0(\mathbb{R}^d)$ corresponding to a probability measure, μ, in $\mathbb{K}(\mathbb{R}^d)$ is mark-locally finite if and only if the measure $\rho_{\mu_{\mathcal{R}^{-1}}}$ in $\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)$ is locally finite.*

Proof. Let ρ_μ be the correlation measure of a measure, μ, in $\mathbb{K}(\mathbb{R}^d)$. Then, for a set $A \in \mathcal{B}_{\mathrm{cm}}(\mathbb{K}_0)$,

$$\begin{aligned}\rho_\mu(A) &= \int\limits_{\mathbb{K}(\mathbb{R}^d)} \mathcal{K}(A,\eta)\mu(d\eta) = \int\limits_{\mathbb{K}(\mathbb{R}^d)} (K_{\mathbb{K}}\mathbb{1}_A)(\eta)\mu(d\eta) \\ &= \int\limits_{\mathbb{K}(\mathbb{R}^d)} \left[K_\Pi(\mathcal{R}^{-1}\mathbb{1}_A)\right](\mathcal{R}^{-1}\eta)\mu(d\eta) = \int\limits_{\mathbb{K}(\mathbb{R}^d)} [K_\Pi \mathbb{1}_{\mathcal{R}^{-1}A}](\mathcal{R}^{-1}\eta)\mu(d\eta) \\ &= \int\limits_{\Pi(\mathbb{R}_0^d\times\mathbb{R}^d)} (K_\Pi \mathbb{1}_{\mathcal{R}^{-1}A})(\gamma)\mu_{\mathcal{R}^{-1}}(d\gamma) \\ &= \rho_{\mu_{\mathcal{R}^{-1}}}(\mathcal{R}^{-1}A),\end{aligned}$$

where $\mu_{\mathcal{R}^{-1}}$ is the pullback measure of μ under $\mathcal{R}$. Reversing the calculations yields the converse result. □

Mark-local absolute continuity allows us to compare measures in $\mathbb{K}(\mathbb{R}^d)$ and $\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)$. The following Lemma will be used to compare measures in $\mathbb{K}(\mathbb{R}^d)$ and $\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)$.

Lemma 10. *Let μ, as a probability measure in $\mathbb{K}(\mathbb{R}^d)$, be mark-locally absolutely continuous concerning μ_λ. Then, $\mu_{\mathcal{R}^{-1}}$ in $\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)$ is locally absolutely continuous with respect to π_σ.*

Proposition 16. *Suppose μ is a probability measure in $\mathbb{K}(\mathbb{R}^d)$ that satisfies the following conditions; it is mark-locally absolutely continuous concerning μ_λ, and its associated correlation measure ρ_μ is mark-locally finite. Then, we can conclude that $\mu_{\mathcal{R}^{-1}} \in \mathcal{M}^1_{\mathrm{fm}}(\Pi(\mathbb{R}^d_0 \times \mathbb{R}^d))$, and it is locally absolutely continuous with respect to π_σ. Additionally, we can observe that $\rho_{\mu_{\mathcal{R}^{-1}}}$ is absolutely continuous with respect to $\mathcal{L}_\sigma$ and its correlation function is well defined.*

Proof. The proof follows from Proposition 15, Lemma 10, and Proposition 7. □

The results obtained above imply the existence of a density function.

Theorem 3. *We assume the conditions of Proposition 16. Then, the correlation function of μ exists, i.e., a function $k_\mu \colon \mathbb{K}_0(\mathbb{R}^d) \to \mathbb{R}$ such that k_μ is the density function of ρ_μ with respect to $\mathcal{L}_\sigma$.*

Proof. By the Proposition 16, we obtain the existence of a correlation function for $\rho_{\mu_{\mathcal{R}^{-1}}}$ in $\Pi_0(\mathbb{R}^d_0 \times \mathbb{R}^d)$ with respect to $\mathcal{L}_\sigma$. We obtain the desired result by transferring this function by the reflection mapping $\mathcal{R}$. □

From a mathematical perspective, it is beneficial to represent a hierarchical structure associated with a function, $k \colon \mathbb{K}_0(\mathbb{R}^d) \to \mathbb{R}$. This representation allows the function in $\mathbb{K}_0(\mathbb{R}^d)$ to be expressed as a sequence of functions in $(\mathbb{R}^d_0 \times \mathbb{R}^d)^n$. Consequently, an evolution equation defined in an infinite-dimensional space can be reformulated in practical applications as a sequence of evolution equations in finite-dimensional spaces.

Definition 20. *Let $k \colon \mathbb{K}_0(\mathbb{R}^d) \to \mathbb{R}$. The hierarchical structure corresponding to k is defined as the sequence of symmetric functions $\{k^{(n)}\}_{n=0}^\infty$, $k^{(n)} \colon (\mathbb{R}^d_0 \times \mathbb{R}^d)^n \to \mathbb{R}$ by the following:*

$$k^{(n)}(v_1, x_1, \ldots, v_n, x_n) := \begin{cases} k(\sum_{i=1}^n v_i \delta_{x_i}), & \text{if } \eta = \sum_{i=1}^n v_i \delta_{x_i} \in \mathbb{K}_0^{(n)}(\mathbb{R}^d), \\ 0, & \text{otherwise.} \end{cases}$$

The function $k_\mu^{(n)}$ is referred to as the n-point correlation function of μ. For convenience, we also write the following:

$$k^{(n)}(v_1, \ldots, x_n) := k^{(n)}(v_1, x_1, \ldots, v_n, x_n).$$

We define the following notion of a correlation function:

Definition 21. *The n-point correlation function in $\mathbb{K}_0(\mathbb{R}^d)$ with respect to positions is defined as follows:*

$$\varkappa_{\mu,h}^{(n)}(x_1, \ldots, x_n) := \int_{(\mathbb{R}^d_0)^n} < h, v_1 > \cdots < h, v_n > k_\mu^{(n)}(v_1, \ldots, x_n) \lambda(dv_1) \ldots \lambda(dv_n),$$

where $k^{(n)}$ is the n-point correlation function introduced in Definition 20 and $h \in \mathbb{R}^d$.

Remark 2. *The function $\varkappa^{(n)}$ can be obtained by the integration of $k^{(n)}$; therefore, we can proceed by only analyzing the latter.*

3. Discussion

We developed a rigorous mathematical framework to analyze classical continuous systems with singular velocity distributions, addressing a significant gap in modeling such systems. These distributions, represented by Radon measures with infinite mass, introduce challenges beyond standard approaches in configuration spaces. In particular, we focused on constructing and analyzing the phase space of configurations and measures, with specific emphasis on the Plato space $\Pi(\mathbb{R}^d_0 \times \mathbb{R}^d)$ and the cone of vector-valued

discrete Radon measures $\mathbb{K}(\mathbb{R}^d)$, connected via the reflection mapping $\mathcal{R}$. This relationship is the foundation for developing harmonic analysis and studying associated measures and dynamics. Let us underline that the conceptualization of the Plato space introduces an idealized setting where configurations are locally finite subsets of $\mathbb{R}_0^d \times \mathbb{R}^d$, which are required to satisfy the condition that for any compact subset $\Lambda \subset \mathbb{R}^d$, the total velocity of particles within Λ remains finite, expressed as follows:

$$V_\Lambda(\gamma) = \sum_{x \in \tau(\gamma) \cap \Lambda} |v_x| < \infty, \quad \forall \gamma \in \Pi(\mathbb{R}_0^d \times \mathbb{R}^d),$$

with $\tau(\gamma)$ being the projection of the configuration γ onto spatial positions. This constraint ensures that the system avoids unbounded velocities within finite spatial domains, aligning with physical plausibility. Moreover, since the reflection mapping $\mathcal{R}$ connects the Plato space $\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)$ with the cone $\mathbb{K}(\mathbb{R}^d)$ by the mapping configurations $\gamma \in \Pi(\mathbb{R}_0^d \times \mathbb{R}^d)$ to vector-valued Radon measures,

$$\mathcal{R}\gamma = \sum_{x \in \tau(\gamma)} v_x \delta_x, \quad \text{where } v_x \in \mathbb{R}_0^d,$$

then it defines $\mathbb{K}(\mathbb{R}^d)$ as the image of $\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)$ under $\mathcal{R}$. The latter gives the possibility of forming the space of observable objects derived from ideal configurations. Consequently, the topology of $\mathbb{K}(\mathbb{R}^d)$ inherits its structure from the vague topology in the configuration space. In the latter setting, a cornerstone of the analysis lies in the use of Poisson measures in $\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)$, defined via their Laplace transform:

$$\int\limits_{\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)} \exp\left(\sum_{(v,x) \in \gamma} \psi(v,x) \right) \pi_\sigma(d\gamma) = \exp \int\limits_{\mathbb{R}_0^d \times \mathbb{R}^d} \Big(e^{\psi(v,x)} - 1 \Big) \sigma(dv, dx),$$

where $\sigma = \lambda \otimes m$ is an intensity measure combining a Radon measure $\lambda(dv)$ in $\mathbb{R}_0^d$ and a Lebesgue measure $m(dx)$ in $\mathbb{R}^d$, hence encapsulating the statistical properties of the system and forms the basis for deriving probability measures in $\mathbb{K}(\mathbb{R}^d)$. We also implemented the associated harmonic analysis via the K-transform to facilitate the transition between finite configurations in $\Pi_0(\mathbb{R}_0^d \times \mathbb{R}^d)$ and infinite configurations in $\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)$.

Indeed, this allows the extension of functions and measures from finite to infinite configurations, providing a powerful tool for analyzing a system's dynamics. Moreover, introducing the measures in $\mathbb{K}(\mathbb{R}^d)$ via the pushforward of measures in $\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)$ under $\mathcal{R}$ allowed us to study complex systems with singular velocity distributions. Accordingly, we derive a central result of the paper, i.e., we proved that the image σ-algebra of $\Pi(\mathbb{R}_0^d \times \mathbb{R}^d)$ under $\mathcal{R}$ coincides with the σ-algebra of $\mathbb{K}(\mathbb{R}^d)$, ensuring the measurability of the reflection mapping and the compatibility of the mathematical structures. The results discussed above notably extend the scope of prior research on configuration spaces and Radon measures, especially those concentrating on positive measures in Riemannian contexts. Classical studies, such as those by Kondratiev et al., have explored the harmonic analysis of positive discrete Radon measures in Riemannian manifolds, representing particle configurations with purely spatial characteristics. These configurations are typically modeled as locally finite sets with well-behaved measures, facilitating the development of analytical techniques such as correlation functions and generating functionals. In contrast, the framework developed here addresses a broader and more complex class of systems: vector-valued Radon measures with singular distributions. These singularities arise when measures have an infinite mass or exhibit significant anisotropy in velocity space. Specifically, velocity distributions of the form $\lambda(dv) = |v|^{-\alpha} e^{-|v|^\beta} dv$, with $\alpha \in [d, d+1)$ and $\beta > 0$, present deep mathematical and physical consequences. These distributions enable the inclusion of systems where velocities can become arbitrarily large near specific phase space domains, a scenario that cannot be described using traditional finite measures. In particular, the

parameter α determines the degree of singularity near $v = 0$. For $\alpha = d$, the measure $\lambda(dv)$ transitions to a marginally finite scenario. In contrast, values of α approaching $d + 1$ emphasize the singular nature of the distribution, leading to configurations that challenge standard assumptions of local finiteness. On the other hand, the parameter β controls the distribution's decay rate at large $|v|$, with larger values corresponding to faster decay. The interplay between α and β broadens the modeling flexibility, accommodating a wide variety of physical systems; for example, when $\beta = 2$, the measure approximates modified Maxwellian distributions relevant in statistical physics, while for $\beta < 2$, the distributions model systems with heavy-tailed velocities characteristic of certain astrophysical or turbulent systems.

The shift from positive measures to vector-valued measures introduces additional complexity due to the need to account for the vectorial nature of the marks (velocities) attached to spatial configurations. This extension requires new mathematical tools to handle the coupled spatial and velocity dependencies. For instance, the harmonic analysis developed for positive measures relies on scalar-valued generating functions and correlation measures. In contrast, vector-valued measures necessitate functionals and transforms that respect the vector structure. Let us also underline that the introduction of singular velocity distributions also enhances the physical realism of the model, enabling the study of systems where unbounded velocities naturally occur, such as in the following: Astrophysical systems, where stars and galaxies often exhibit velocity distributions with heavy tails due to gravitational interactions. The singular measures proposed here can model such systems without artificial truncations of velocity magnitudes. Kinetic theory: namely in the Boltzmann framework, velocity distributions often develop singular features due to collision dynamics, especially in dilute gasses. Turbulence and fluids: here, singular velocity distributions appear in the statistical description of turbulent flows, where small-scale dynamics generate extreme velocities.

The generalizations discussed above introduce additional complexity in defining and analysing measures in configuration spaces. Specifically, the transition from scalar-valued to vector-valued measures in $\mathbb{K}(\mathbb{R}^d)$ requires the careful consideration of both topological and algebraic structures. Key to this is ensuring the measurability of the reflection mapping $\mathcal{R} : \Pi(\mathbb{R}_0^d \times \mathbb{R}^d) \to \mathbb{K}(\mathbb{R}^d)$ and the compatibility of the corresponding σ-algebras. The K-transform, originally developed for scalar measures, is extended here to accommodate vector-valued measures, providing robust harmonic analysis and probabilistic modeling. This extension enables the handling of singular and anisotropic velocity distributions, thus broadening the framework's applicability to various physical and mathematical problems. This includes exploring systems where classical assumptions, such as bounded velocities or locally finite configurations, do not hold, opening new research directions in studying complex systems.

Future work will aim to extend this framework to interacting particle systems, where pairwise or higher-order interactions introduce additional complexities. Additionally, generalizing the framework to non-Euclidean geometries, such as Riemannian or hyperbolic spaces, will shed light on systems with curvature-dependent dynamics. Numerical implementations of the K-transform and associated measures could facilitate the application of these methods to real-world systems. Lastly, investigating stochastic evolution in $\mathbb{K}(\mathbb{R}^d)$ in the context of (stochastic) mean-field games is a promising direction, linking the theoretical framework to dynamic processes.

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